



József Mezei

# A Quantitative View on Fuzzy Numbers

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# Abstract

Since its introduction, fuzzy set theory has become a useful tool in the mathematical modelling of problems in Operations Research and many other fields. The number of applications is growing continuously. In this thesis we investigate a special type of fuzzy set, namely fuzzy numbers. Fuzzy numbers (which will be considered in the thesis as possibility distributions) have been widely used in quantitative analysis in recent decades.

In this work two measures of interactivity are defined for fuzzy numbers, the possibilistic correlation and correlation ratio. We focus on both the theoretical and practical applications of these new indices. The approach is based on the level-sets of the fuzzy numbers and on the concept of the joint distribution of marginal possibility distributions. The measures possess similar properties to the corresponding probabilistic correlation and correlation ratio. The connections to real life decision making problems are emphasized focusing on the financial applications.

We extend the definitions of possibilistic mean value, variance, covariance and correlation to quasi fuzzy numbers and prove necessary and sufficient conditions for the finiteness of possibilistic mean value and variance. The connection between the concepts of probabilistic and possibilistic correlation is investigated using an exponential distribution.

The use of fuzzy numbers in practical applications is demonstrated by the Fuzzy Pay-Off method. This model for real option valuation is based on findings from earlier real option valuation models. We illustrate the use of number of different types of fuzzy numbers and mean value concepts with the method and provide a real life application.



# Sammanfattning

Diffus mängdlära (eng. “fuzzy set theory”) har blivit ett användbart verktyg för matematisk modellering av problem inom operationsanalys samt även inom många andra områden. Antalet tillämpningar ökar kontinuerligt. I denna avhandling undersöks en speciell typ av diffusa mängder, det vill säga diffusa tal (eng. “fuzzy numbers”). Diffusa tal (som i avhandlingen kommer att beaktas som möjlighetsdistributioner, eng. “possibility distributions”) har använts i stor utsträckning inom kvantitativ analys under de senaste decennierna.

I detta arbete kommer två mått av interaktivitet att definieras för diffusa tal, möjlighets (eng. “possibilistic”) korrelation och korrelationsförhållande. Vi fokuserar både på den teoretiska och praktiska tillämpningen av dessa nya index. Metoden bygger på redan etablerade nivå-uppsättningar av diffusa tal och på ett begrepp gällande marginella möjlighetsdistributioner. Dessa har liknande egenskaper som motsvarande “traditionella” sannolikhetskorrelationer. Genom att fokusera på finansiella tillämpningar granskar vi praktiska tillämpningsmöjligheter inom beslutsfattandeprocessen.

Vi bygger vidare på definitionerna av möjliga (eng. “possibilistic”) medelvärden, varians, kovarians och korrelation till kvasi diffusa tal och vi stipulerar nödvändiga och tillräckliga villkor för ändlighet av möjlighets-medelvärde och varians. Sambandet mellan begreppen sannolikhets- och möjlighetskorrelation utreds med hjälp av en exponentiell fördelning.

Vi demonstrerar användningen av diffusa tal i praktiska tillämpningar av den så kallade “fuzzy pay-off” metoden. Denna modell för värdering av olika handlingsalternativ är baserad på resultat från tidigare värderingsmodeller för realoptioner. Vi illustrerar också användandet av den nya metoden genom praktiska tillämpningar.





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Åbo, November, 2011.



# List of original publications

1. Christer Carlsson, Robert Fullér and József Mezei. A Quantitative Approach to Quasi Fuzzy Numbers, IFSA World Congress, June 21-25, 2011, Surabaya, Indonesia, FN-001-1-FN-001-6.
2. Mikael Collan, Robert Fullér and József Mezei. A Fuzzy Pay-off Method for Real Option Valuation, *Journal of Applied Mathematics and Decision Sciences*, 2009(2009), 1-14.
3. Mikael Collan, Robert Fullér and József Mezei. A Fuzzy Pay-Off Method for Real Option Valuation: Credibilistic Approach, in: Y. Gunalay and S. Kadipasaoglu, S. eds., *Proceedings of the 3rd International Conference on Applied Operational Research - ICAOR 2011*, Istanbul, Turkey, 267-276.
4. Robert Fullér, József Mezei and Péter Várlaki. An improved index of interactivity for fuzzy numbers, *Fuzzy Sets and Systems*, 165(2011), 56-66.
5. Robert Fullér, József Mezei and Péter Várlaki. A Correlation Ratio for Possibility Distributions, in: E. Hüllermeier, R. Kruse, and F. Hoffmann eds., *Proceedings of the International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2010)*, June 28 - July 2, 2010, Dortmund, Germany, Lecture Notes in Artificial Intelligence, 6178(2010), Springer-Verlag, Berlin Heidelberg, 178-187.
6. Robert Fullér, József Mezei and Péter Várlaki. Some Examples of Computing the Possibilistic Correlation Coefficient from Joint Possibility Distributions, in: Imre J. Rudas, János Fodor, Janusz Kacprzyk eds., *Computational Intelligence in Engineering, Studies in Computational Intelligence Series*, 313(2010), Springer, 153-169.



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## Part I

# Research summary





# Chapter 1

## Introduction

The role of uncertainty is inherent in the mathematical modelling of decision-making problems and in their applications. The inevitable presence of uncertainty in complex environments facilitates the development of new techniques to aid the decision makers in risk assessment and mitigation. There exist several definitions of uncertainty in the literature so it is important to clarify what we mean by it. In [1] uncertainty is defined as a situation in which one has no (or limited) knowledge about which of several states of nature has occurred or will occur. This definition highlights the observation that uncertainty is not only present in future events but it also pertains to the analysis of the past.

To examine this from a different perspective, it is evident that uncertain variables should be classified at least into two classes [24]:

- subject to intrinsic variability (randomness),
- totally deterministic but anyway ill-known, either because they pertain to the future, or because of the lack of knowledge (incomplete information).

Since in these two cases the sources of uncertainty are different by nature, the appropriate mathematical models to handle different settings have to be chosen carefully [47].

In the class of problems subject to randomness, the traditional choice is that of using probability theory based methods (stochastic processes, statistical estimations, queueing theory). The reliability of these models when facing incomplete information is not straightforward. The theory of fuzzy sets was introduced in 1965 by Zadeh [106] as a tool to represent and analyse quantities which possess uncertain features different from randomness, and later [111] possibility theory was developed based on fuzzy sets. Since the introduction of fuzzy sets into decision analysis (one of the first papers in this direction was the influential contribution of Bellman and Zadeh [2]), this theory was proved to be an efficient tool to manage problems when facing incomplete knowledge and an alternative model to probability theory in situations when its use is not well-grounded.

In a recent paper [25], Dubois specified the main application areas of fuzzy set theory in decision sciences:

- Gradual preference relations [37].
- Multicriteria aggregation techniques [42].
- Fuzzy interval computations to cope with uncertainty in numerical aggregation [97].
- Fuzzy interval comparison to choose between alternatives with fuzzy ratings [96].
- Linguistic variables to model human-originated information [108, 109, 110].

## 1.1 Quantitative view on fuzzy numbers

One of the most attractive features of probability theory is the existence of some properly defined normative measures. As the concepts of mean value, variance, covariance and correlation coefficient are well-known and extensively used measures of different characteristics of probability distributions, it has been an important research question since the introduction of fuzzy sets and possibility theory, to formulate the corresponding definitions for possibility distributions (if they are meaningful in this context). As one of the first systematic approaches, it is important to mention the *interval-valued mean value* of a fuzzy number [28], which is based on random sets.

In 2001, Carlsson and Fullér defined the possibilistic mean value and variance of a fuzzy number [9] and their definition only uses the concepts of possibility theory independently of probabilities. These concepts have been studied and applied in many fields since their introduction. In the following years, Carlsson, Fullér and Majlender introduced the notion of possibilistic covariance and correlation [13, 39] using the same approach. Although the interaction of possibility distributions has already been defined in [108, 109, 110], this novel definitions of covariance and correlation provide a tool to obtain proper estimations concerning the relationships of the variables in a complex model which is essential in many applications.

**Example 1.1.** *Interactivity plays a very fundamental role in financial mathematics and portfolio optimization. Suppose, an investor has an existing portfolio of stocks and he/she has an option to choose between two new assets which can be included in the portfolio. Assuming that the expected return will be the same whatever option is chosen, how can it be decided, which one to invest in? From the general theory of portfolio optimization, it is known that in order to minimize the risk, the best choice is the portfolio with the minimum aggregated variance. To estimate the variance, it is naturally essential to utilize a measure which can provide a reliable assessment of risk. Without appropriate measures, the identification of promising business opportunities becomes very difficult.*

## 1.2 Fuzzy logic in real life decisions

The standard process of handling uncertainty in real life decision making can be summarized in 3 main steps [65]:

- Reduce uncertainty by a thorough information search,
- Quantify the residue that cannot be reduced,
- Plug the result into some formal scheme that incorporates uncertainty as a factor in the selection of a precourse of action.

In line with the previously mentioned distinction of different types of uncertainties, Koopmans [52] classifies them into two groups: (a) environment related uncertainties, which are unpredictable, and the decision maker has no (or hardly any) means to prevent them; (b) uncertainties related to the inaccuracy of the decision maker which is rooted in the lack of appropriate knowledge or simply in wrong judgements. The uncertainties belonging to the second group are highly related to the behavioural aspects of decision making and cannot be described by a model which builds on the intrinsic variability of events. These are the cases when fuzzy set theory can provide a useful tool.

When the traditionally used crisp estimations are replaced by fuzzy numbers, we can express in the models the inaccuracy of human perception. This approach also applies to the cases when the existing historical data is not reliable enough to be the basis of the estimation of future events (in stock markets, unexpected events happen more and more frequently [91]), or there is no historical data at all (real options are usually based on investments which cannot be observed in real markets). This makes these models more realistic, as they do not simplify uncertain distribution-like observations to a single point estimate that conveys the sensation of no-uncertainty (like the net present value of an asset).

## 1.3 Research problems and the structure of the thesis

The dissertation investigates three main problems:

- interactivity measures of fuzzy numbers,
- generalization of characteristic measures to quasi fuzzy numbers,
- the application of fuzzy numbers in real world decisions, specifically in real option analysis.

Chapters 2 and 3 provide the research methodology and the basic definitions required for the detailed analysis, respectively. In this section, we shortly summarize the contributions from the different chapters and specify the connections to the published papers.

## Interactivity measures

A measure of possibilistic correlation between fuzzy numbers  $A$  and  $B$  has been defined in [13] as their possibilistic covariance divided by the square root of the product of their possibilistic variances. In this paper the authors proved that the correlation coefficient takes its value from the  $[-1, 1]$  interval if the level-sets of the joint distribution are convex. In Chapter 4 of this thesis, an example is shown to illustrate that this property does not necessarily hold when the level-sets are non-convex. After this a new index of interactivity is defined for fuzzy numbers which always takes its value from the unit interval and possesses similar properties to the probabilistic correlation coefficient. This can be seen as a new/alternative possibilistic correlation. The behaviour of this new concept is illustrated through a series of examples describing the most important and commonly used joint distributions. This chapter is supported by the following original publications:

- Robert Fullér, József Mezei and Péter Várlaki. An improved index of interactivity for fuzzy numbers, *Fuzzy Sets and Systems* 165 (2011), 56-66.
- Robert Fullér, József Mezei and Péter Várlaki. Some Examples of Computing the Possibilistic Correlation Coefficient from Joint Possibility Distributions, in: Imre J. Rudas, János Fodor, Janusz Kacprzyk eds., *Computational Intelligence in Engineering, Studies in Computational Intelligence Series*, 313(2010), Springer, 153-169.

To answer the question whether it is possible to define a different interactivity measure for fuzzy numbers which reflects different properties of the marginal distributions than the correlation, in Chapter 5 the correlation ratio of fuzzy numbers is defined based on the definition of the probabilistic correlation ratio. This new notion is not a simple modification of the correlation, it describes a different point of view on the relationships between possibility distributions. When one has to deal with a problem of great complexity and a very complicated structure of interaction between the variables, it is always advisable to examine it from different perspectives, and this new definition provides an alternative tool to analyse the sensitivity of decisions (for example in a portfolio optimization problem in possibilistic settings). This chapter is based on the original paper:

- Robert Fullér, József Mezei and Péter Várlaki. A Correlation Ratio for Possibility Distributions, in: E. Hüllermeier, R. Kruse, and F. Hoffmann eds., *Proceedings of the International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2010)*, June 28 - July 2, 2010, Dortmund, Germany, Lecture Notes in Artificial Intelligence, 6178(2010), Springer-Verlag, Berlin Heidelberg, 178-187.

## Quasi fuzzy numbers

In most of the applications, although theoretically the final result can take its value anywhere in the set of real numbers, in practice the tails of the distributions are truncated to obtain a fuzzy number with bounded support (in most of the cases the simplest distribution, a trapezoidal or triangular fuzzy number). This is common in possibilistic modelling, not in statistics. Recent years in the financial markets have shown that it is not possible to overestimate the effect of specific events [91]. In these cases the use of fuzzy numbers with infinite tail (quasi fuzzy numbers) can provide a solution. In Chapter 6 the problem of the generalization of the characteristic measures to quasi fuzzy numbers is investigated and a subset of them is identified which possesses certain properties (finiteness of mean value and variance). The following original paper contributes to this chapter:

- Christer Carlsson, Robert Fullér and József Mezei. A Quantitative Approach to Quasi Fuzzy Numbers, IFSA World Congress, June 21-25, 2011, Surabaya, Indonesia, FN-001-1-FN-001-6.

## The fuzzy pay-off method

Fuzzy sets are sets that allow gradation of belonging, such as "the value of a future cash flow at year 5 is about 5000". This means that fuzzy sets can be used to formalize inaccuracy that exists in human decision making and as a representation of vague, uncertain or imprecise knowledge, e.g., future cash-flow estimation. Chapter 7 presents a new method for valuation of real options from fuzzy numbers which is based on the previous literature on real option valuation, especially the findings presented in [21] and the model is illustrated by a selection of different types of fuzzy numbers. Two different concepts of mean value for fuzzy numbers are employed and the results are compared. Finally, the method is illustrated by a real world decision making example (valuation of patents). This chapter is based on the original publications

- Mikael Collan, Robert Fullér and József Mezei. A Fuzzy Pay-off Method for Real Option Valuation, *Journal of Applied Mathematics and Decision Sciences*, 2009(2009), 1-14.
- Mikael Collan, Robert Fullér and József Mezei. A Fuzzy Pay-Off Method for Real Option Valuation: Credibilistic Approach, in: Y. Gunalay and S. Kadipasaoglu, S. eds., *Proceedings of the 3rd International Conference on Applied Operational Research - ICAOR 2011*, Istanbul, Turkey, 267-276.



## Chapter 2

# Methodology

This dissertation can be positioned in the intersection of two research fields:

- Operations Research (OR): mathematical modelling has always been an essential part of OR. The aim of these methods is to help organizations and decision makers to find the optimal solution (or a solution which is as close to optimal as it is possible). Probabilistic modelling has played a fundamental role in OR from the beginning and after the introduction of fuzzy sets, possibility theory also became a very popular research topic, particularly in decision analysis.
- Financial mathematics: the theory of option valuation has been a very significant research direction in the last decades. In Chapter 7 real options are considered which are different from financial options by nature. In general, the models used for valuing this type of investments are originated from financial options, not taking into consideration the unique characteristics inherent to real options. Our aim is to propose a model which incorporates the specific properties of real options.

In this chapter first a short description of these two research fields is presented with the focus on the modelling of uncertainty in OR and finance. The research philosophies adapted in the thesis are specified and finally a section is provided on the nature of probabilistic and possibilistic modelling.

### 2.1 Operations Research

Operations Research is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis [85, 73]. Operations Research was formally developed during the Second World War to plan and organize military operations in a more efficient way than ever before. Since its introduction, the scope of Operations Research has been extended to a wide range of fields and the developed mathematical models became indispensable in the everyday activities of organizations and provide an efficient way to solve various problems (inventory

analysis, project management, resource allocation, routing, scheduling). The interactive process of problem solving through Operational Research can be summarized in the following 3 steps:

1. The identification of all the possible alternative solutions for the problem.
2. Using a mathematical model (and taking into account the preferences of the decision maker), the aim is to find the optimal solution if it is possible. If the complexity of the problem makes it impossible to identify the optimal solution in an acceptable timeframe, heuristic methods can be employed to find a satisfactory solution.
3. The decision maker reflects his opinion about the solutions found, and if one of them is accepted, it can be tested in a real life situation. If the derived alternatives do not meet the requirements of the decision maker, a new solution has to be found by refining the model based on new specifications.

In order to provide an efficient and useful tool for organizations, a mathematical model has to possess 3 essential properties [43]: understandable, verifiable and reproducible. When modelling an operational/decision-making problem, first we need to understand a set of activities constituting a complex system, then utilize this knowledge to predict or improve the performance of the system in a verifiable way, and this process should be reproducible.

The modelling of uncertainty plays a fundamental role in Operations Research and specifically in decision making. Uncertainty can emerge from various sources [49]:

- act-event sequences,
- event-event sequences,
- value of consequences,
- appropriate decision processes,
- future preferences and actions,
- one's (in)ability to affect future events.

This thesis is mostly concerned with two of these problems: (a) appropriate decision processes require appropriate and reliable decision models (the construction of normative measures to quantify the available knowledge in a processable and useful way); (b) value of consequences: before a company takes on an investment opportunity, a reliable estimation of the value of this investment is required, even in situations when there is not enough historical data available to employ for example statistical models. The methods in this case should be built on the subjective judgements of the decision makers.



## Multiobjective optimization

The general purpose of multiobjective/multicriteria optimization [32] is to find a compromise when there are several (more than one) objectives present, which conflict each other. It is required in practical applications that the offered solution should be the best-fit to the needs of the decision maker. This compromise between conflicting objectives is termed as an optimality principle. Contrary to the single objective case, the optimality in multicriteria optimization can be defined in various ways, for example Pareto or proper Pareto optimality, weak efficiency, lexicographic optimality. The choice of the optimality principle in a given problem depends on the type of the solution required and specified by the decision maker. Optimization with multiple objectives also appeared in the context of fuzzy set theory [114].

A general multicriteria optimization problem has the following form

$$\min_{x \in S} \{f_1(x), f_2(x), \dots, f_k(x)\},$$

where  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  are the objective functions.  $S \subset \mathbb{R}^n$  is the set of all possible solutions (feasible set in the decision space). The points in the image of the feasible set,  $Z \subset \mathbb{R}^k$ , are termed objective vectors and denoted by  $z = f(x) = (f_1(x), f_2(x), \dots, f_k(x))^T$ . The most commonly used optimality principle is the following:

**Definition 2.1** ([32]). *An objective vector  $z^* \in Z$  is Pareto optimal or efficient if there does not exist another objective vector  $z \in Z$  such that  $z_i \leq z_i^*$  for all  $i \in I_k$  and  $z_j < z_j^*$  for at least one index  $j$ .*

**Example 2.1.** *A portfolio selection problem can be formalized in terms of a bi-objective optimization problem: the two objectives are the expected return and the variance of the portfolio. Assuming the risk aversion of the investors, the expected return has to be maximized and simultaneously the variance has to be minimized. If 3 portfolios are considered with returns and variances listed in Table 2.1, when employing the Pareto*

	Return	Variance
$P_1$	100	30
$P_2$	120	35
$P_3$	80	40

Table 2.1: Simple example for portfolio selection

*principle it can be seen that  $P_1$  and  $P_2$  dominate  $P_3$ , but this rule does not help to decide between the first 2 portfolios: in terms of return  $P_2$  is better, but  $P_1$  offers smaller variance.*

To handle the situations when there are several non-dominated solutions present (as in the previous example), the usage of trade-offs is suggested. If the decision maker

is able to quantify his preferences, for example: 10 units of increase in the return and 5 units of decrease in the variance can be considered equivalent to him, then in the previous example,  $P_2$  is the portfolio with the best performance. In [46], the concept of trade-offs has been formulated in terms of a cone of trade-off directions. This approach is investigated further by Mäkelä, Nikulin and Mezei in [77, 76].

## 2.2 Mathematics in finance

Mathematical finance aims at constructing formal mathematical models to help decision makers in the financial markets. “The secret of success in financial management is to increase value” [6]: this can be facilitated by finding the appropriate answers to two crucial problems:

1. Which investment opportunities should be undertaken by the company?
2. What is the exact amount of capital to be invested in the carefully chosen assets/projects?

One of the simplest, but most commonly used methods to estimate the value of an asset is the net present value (NPV) [36] which is the total present value of a time series of cash flows. It is a standard method for using the time value of money to value long-term projects. The cash inflow/outflow for every time period is discounted back to its present value and the value of the investment is simply the sum of these discounted cash-flows:

$$NPV = \sum_{t=0}^T \frac{C_t}{(1+r)^t},$$

where  $r$  is the discount rate and  $C_t$  is the net cash flow at time  $t$ . In this thesis the main focus is on the pricing of (real) options which cannot be valued by this simple method.

### Option pricing

A financial option is the right - but not the obligation - to engage in a future transaction concerning a financial asset (or assets) at a price fixed in the contract between the buyer and the seller.

- A call option gives the buyer of the option the right to buy the underlying asset at a fixed price (strike price) at the expiration date or at any time prior to the expiration date (European or American call options, respectively).
- A put option gives the buyer of the option the right to sell the underlying asset at a fixed price at the expiration date or at any time prior to the expiration date (European or American put options).

**Example 2.2.** *In plain words it means that if an investor buys a call option, he/she will exercise it at the expiration date (in case of the European option) if the price of the underlying stock in the market is higher than the strike price and this difference is large enough to compensate for the price of the option in order to realize profit from the transaction. In case of a put option, the market price has to be lower than the strike price to realize the profit.*

The difference between the European and American options lies in the expiration date: the American option can be exercised anytime before the expiration. This difference makes it more difficult to calculate the price of an American option while there exist several pricing formulas to calculate the value of European options. It is important to note that because of the possibility to exercise the option anytime before expiration, the price of an American option cannot be lower than the corresponding European option on the same underlying asset.

There exist several ways to determine the value of a financial option, the three most commonly used models originate from the 1970s:

- The Black-Scholes model [3]: the value of the option is described by a partial differential equation and the solution is based on the existence of a perfect hedging.
- Binomial model [20]: a discrete time model in which the value of the option at a timepoint corresponds to a node in a binomial tree. It is important to note that the solution of this model approximates the solution derived from the Black-Scholes equation.
- Monte-Carlo model [4]: the price of the option is obtained as the discounted average of “possible” values which are generated by simulations of the process.

Since its introduction, the Black-Scholes model became the most widely used valuation method. It is worthwhile to recall the assumptions which are necessary in order to obtain the price of the option through the partial differential equation [3]:

- There is a constant and known interest rate.
- The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. The variance of the underlying asset is constant.
- There are no dividends or transaction costs.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.
- There are no penalties to short selling.

To formulate the price of a European call option obtained from the Black-Scholes model, the following notations will be used:  $S$  is the price of the stock,  $C(x, t)$  is the price of a European call option at time  $t$ ,  $c$  is the strike price of the option,  $r$  is the annualized risk-free interest rate,  $\sigma$  is the volatility of the stock's returns and  $T$  is the date of expiry. Then the value of the option is:

$$C(x, t) = xN(d_1) - ce^{r(t-T)}N(d_2),$$

where  $N$  denotes the standard normal cumulative distribution function,

$$d_1 = \frac{\ln(\frac{x}{c}) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$

and

$$d_2 = \frac{\ln(\frac{x}{c}) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} = d_1 - \sigma\sqrt{T - t}.$$

A similar formula can be derived for a call option.

In [11], Carlsson and Fullér introduced the possibilistic version of the Black-Scholes model where the price of the stock and the strike price are represented by trapezoidal fuzzy numbers. In the paper, the model is used to determine the value of a real option.

The term real option was introduced in [75] and it refers to the valuation of options where the underlying asset is non-financial. For example, a real life investment or an R&D project can be seen as an option and modelled by using the theory of traditional financial assets. The main factors, where a real option differs from a financial one, are the following [5, 94, 90]: the maturity is always longer; the value can be increased by managerial decisions; large value; the value depends on the competition; real options are not traded (there exists no market of real investments). These fundamental differences raise the question of the applicability of the traditional valuation methods in the context of real options. This problem will be further investigated in Chapter 7.

## 2.3 Philosophical aspects

As this thesis is concerned with mathematical methods and the application of the models to real life practical problems, the most important philosophy underlying the research process is positivism. The principles of positivism were described by August Comte in the 19th century for the first time. One of the main components of positivism is the scientific method which replaces the metaphysics in all sciences, thus claiming that the logic behind every type of particular research should be the same, independently of the phenomenon.

In particular in mathematics, the seeds of positivism can be observed starting from the ancient Greek scientists. One of the most important contributors of positivism in mathematical sciences was Bertrand Russell, who published his monumental work, the *Principia Mathematica* co-written by Alfred North Whitehead in the 1910's. The aim

of this book is to show that every mathematical statement can be derived from a small set of axioms using only mathematical logic. The ideas of Russell's book (together with Ludwig Wittgenstein's *Tractatus Logico-Philosophicus*) had significant influence on the development of the logical positivism. This movement can be characterized by two main propositions:

- Verifiability: a statement is meaningful, if it can be verified in finite number of steps
- Unified Science: not only the driving methodology, but a unified scientific language in all the sciences

Although this 'radical' direction of positivism slowly disappeared from the main stream of philosophy of science as some of its ideas proved to be inapplicable in many areas, in developing mathematical models this interpretation can not be overlooked.

Another important principle of positivism is the testability (empirical investigation). This aspect becomes especially crucial when developing mathematical models for financial problems. The last decades have seen many attempts to provide a reasonable mathematical description of financial markets but it would be very difficult to find one of them which would stand against the theory of falsifiability ([91]).

Probably the most influential philosophical approach for my research has been the work of Imre Lakatos. His book, *Proofs and Refutations* [56] describes the progress of mathematics through a series of attempts to prove a theorem. Every step of the process originates from a counterexample to the previous proposal and results in a proof which generalizes and accumulates all the proceeding useful ideas and tries to correct the mistakes (this idea of progress motivated for example the new definition of correlation for fuzzy numbers). Another important contribution of Lakatos is the proper formalization of the definition of a research programme ([57, 58]). He classified research into progressive and degenerate cases and introduced the concept of pseudoscience (which does not make any novel observations or discover previously unknown facts). For example, he reasoned that the neoclassical theory of economics is a pseudoscience. If we examine the development of fuzzy set theory and how it has found its way to practical applications (besides the uncountably many theoretical contributions), it is safe to claim that this research programme proved to be a progressive scientific theory in terms of Lakatos.

## 2.4 Possibilistic and probabilistic modelling

To avoid any confusion about the topic of this thesis, it is important to clarify that although probability and possibility theory will be used extensively, the scope of this thesis is different from the theory of uncertain or fuzzy probabilities. This subject is well-represented by imprecise probabilities (lower and upper probabilities [23], belief functions [87]). It is well-known that every possibility distribution can be investigated as an upper probability: it defines a unique set of admissible probability distributions.

Instead of defining the characteristics of fuzzy numbers by using this set of probability distributions (as in [28]), we use the fundamental structure of level-sets and define a uniform distribution on these sets.

There has been a long lasting debate amongst the researchers in probability and possibility theory about the superiority of one of these directions. I do not think that “the only satisfactory description of uncertainty is probability” [64] or “probability theory should be based on fuzzy logic” [112]. It always depends on the structure of a problem to decide which one of the available tools is more appropriate. To define the set of problems in which possibility theory is maybe more reasonable, we should turn to an article which was written almost 20 years before the introduction of fuzzy sets: if a theory has been developed as a reaction to the lack of appropriate models to solve some specific problems, this can clearly justify its usefulness.

In [101] (the article was analysed in [86]), Weaver mentioned a trichotomy of scientific problems:

- ‘Problems of simplicity’: that physical science before 1900 was largely concerned with.
- ‘Problems of disorganized complexity’: the number of variables is very large, the behaviour of the variables is individually erratic but the system as a whole possesses certain orderly and analyzable average properties.
- ‘Problems of organized complexity’: problems which involve dealing simultaneously with a sizable number of factors which are interrelated into an organic whole; to explain them something more is needed than the mathematics of averages.

The potential application area of possibility theory lies in this region of organized complexity and recent years have seen a series of successful possibilistic models from finance [8] to health care applications [61], not only in the system design, which was the original intention behind the development of fuzzy sets.

## Chapter 3

# Preliminaries

This chapter provides a brief overview of the fundamentals of probability and possibility theory which are necessary for the formulation of the problems described in the following chapters. First the definition of a monotone measure [98] is recalled since probability and possibility measures are special cases of this construct:

**Definition 3.1** (Normalized monotone measure). *Let  $X$  be a non-empty set and  $\mathcal{C}$  any  $\sigma$ -algebra of its subsets, then a set function  $m : \mathcal{C} \rightarrow [0, 1]$  is a normalized monotone measure if it satisfies*

1.  $m(\emptyset) = 0, m(X) = 1$
2.  $A \subseteq B \Rightarrow m(A) \leq m(B) \forall A, B \in \mathcal{C}$ .

**Definition 3.2** (Probability measure). *A probability measure,  $P$ , is an additive normalized monotone measure, i.e.*

$$P(A \cup B) = P(A) + P(B)$$

*for any disjoint subsets  $A$  and  $B$  of the event space.*

Note that probability is sufficient to describe the likelihood of an event thanks to the autoduality property, i.e.  $P(A) = 1 - P(\bar{A})$ .

**Definition 3.3** (Possibility). *A (normalized) possibility measure,  $Pos$ , is a maxitive normalized monotone measure, i.e.*

$$Pos\left(\bigcup_{i \in I} A_i\right) = \sup_i Pos(A_i)$$

*for any family  $\{A_i | A_i \in P(X), i \in I\}$ , where  $I$  is an arbitrary index set.*

But it is not autodual; thus another function is needed:

**Definition 3.4** (Necessity measure). *A necessity function,  $\text{Nec}$ , is a normalized monotone measure defined as*

$$\text{Nec}\left(\bigcap_{i \in I} A_i\right) = \inf_i \text{Nec}(A_i)$$

*for any family  $\{A_i | A_i \in P(X), i \in I\}$ , where  $I$  is an arbitrary index set.*

As it was discussed in the Introduction, probability and possibility measures can represent two different types of uncertainty:

1. Uncertainty due to variability of observations.
2. Uncertainty due to incomplete information.

### 3.1 Probability theory

The mathematics of probability has its roots in the 17th (Pascal) and 18th century (Bernoulli); here modern probability theory is considered which was introduced by Kolmogorov [51], when he developed the formal system of axioms based on measure theory. For a comprehensive description of probability theory and applications, see [33, 34]. Here we are only concerned with continuous distributions: the domain will be denoted by  $\Omega$  and  $F \subset 2^\Omega$  is the set of events. The key concept in probability theory is the notion of a random variable. In the most general form, a random variable is a function

$$X : \Omega \rightarrow M,$$

which is  $(F, G)$ -measurable (where  $(M, G)$  is a measurable space). This property means that for every  $A \in G$  its preimage,  $X^{-1}(A)$  belongs to  $F$ . In this thesis only real-valued random variables are used (in this case  $G$  corresponds to the Borel  $\sigma$ -algebra), it is worth to formulate this property in this special case (using that it is sufficient to check the measurability on a generating set of the  $\sigma$ -algebra):

$$X : \Omega \rightarrow \mathbb{R} \text{ is a random variable} \iff \{\omega \in \Omega \mid X(\omega) \leq t\} \in F, \forall t \in \mathbb{R}.$$

Another important notion is the cumulative distribution function of a random variable which describes the probability distribution of  $X$ :

$$F_X(t) = P(X \leq t), \forall t \in \mathbb{R}.$$

In other words, it determines the probability that the random variable falls within the interval  $[a, b]$ :

$$P(a \leq X \leq b) = F_X(b) - F_X(a).$$

Instead of using the cumulative distribution function to determine probabilities, we will employ the density function,  $f_X$ , of a random variable, which is the derivative of the function  $F_X$ , from which it follows that

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx.$$



Generally, for any arbitrary set,  $A \subseteq \mathbb{R} : P(X \in A) = \int_A f_X(x)dx$ .

In practical problems, we usually deal with families of probability distributions of random variables (for example Gaussian, Gamma). The parameters which can help to identify particular members of these families (and in general play a fundamental role in probability theory) consist of a set of characteristic measures. First of all, it is a very important question, what is the 'average value' of a distribution. This can be captured by the expected value (mean value) of a random variable (since  $E$  will be used for the possibilistic mean value, to avoid any confusion, throughout the thesis  $M$  will stand for the expectation of a probability distribution):

$$M(X) = \int_{\mathbb{R}} x dF_X(x) = \int_{\mathbb{R}} x f_X(x) dx.$$

An important property of the expected value is the linearity.

The next important question after the expected value is to determine the average distance of the outcomes from the average value (in other words the deviation from the expectation). The square of this distance is calculated as

$$\text{var}(X) = M((X - M(X))^2) = M(X^2) - M(X)^2.$$

This measure is termed as the variance of the random variable  $X$ .

**Example 3.1.** *Since the uniform distribution will play a crucial role and will be used in the thesis extensively, it is important to calculate its characteristics. The probability density function of a continuous uniform distribution taking its values from the  $[a, b]$  interval is the following (the uniform distribution will be denoted by  $U$ ):*

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

*Using this, it is easy to calculate the mean value and the variance based on the definitions:*

$$M(U) = \frac{1}{b-a} \int_a^b x dx = \frac{a+b}{2}$$

*and*

$$\text{var}(U) = \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{(b-a)^2}{12}.$$

In most cases, problems cannot be solved simply by examining a single distribution. When there are several variables considered, it is essential to measure the dependencies between distributions. For example, if we want to determine the optimal portfolio of stocks described by random variables, it is necessary to consider the joint effects of particular assets.

Before the definition of different dependency measures, it is necessary to recall the notion of a joint distribution: if  $X$  and  $Y$  are two random variables, then the joint distribution function of  $(X, Y)$  is

$$F_{X,Y}(t, s) = P(X \leq t, Y \leq s), \forall t, s \in \mathbb{R}.$$

As in the one dimensional case, the joint density function,  $f_{X,Y}$ , is the derivative of the distribution function with respect to  $t$  and  $s$ .  $X$  and  $Y$  are called the marginal distributions of the joint distribution. The relation between  $X$  and  $Y$  are completely described by the joint density function, moreover, the marginal density functions can be obtained in the following way:

$$f_X(t) = \int_{\mathbb{R}} f_{X,Y}(t, y) dy, \quad f_Y(s) = \int_{\mathbb{R}} f_{X,Y}(x, s) dx.$$

$X$  and  $Y$  are said to be independent if

$$f_{X,Y}(t, s) = f_X(t)f_Y(s).$$

Using these concepts, it is possible to define different dependency measures for random variables. The covariance of  $X$  and  $Y$  is defined as:

$$\begin{aligned} \text{cov}(X, Y) &= M((X - M(X))(Y - M(Y))) = M(XY) - M(X)M(Y) \\ &= \int_{\mathbb{R}^2} ts f_{X,Y}(t, s) dt ds - \left( \int_{\mathbb{R}} t f_X(t) dt \right) \left( \int_{\mathbb{R}} s f_Y(s) ds \right). \end{aligned}$$

The covariance is an absolute measure of the relationship between the variables. The value of the covariance indicates how much the variables change together: if it is greater than 0 then the two variables tend to increase or decrease together; if it is smaller than 0 then an increase in one of them implies a decrease in the other one in average. It is important to note, that the covariance of independent variables is 0, but 0 covariance does not imply independence.

To define a relative measure of dependency (interactivity), the correlation coefficient was introduced:

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}},$$

where  $X$  and  $Y$  are random variables with non-zero variances. It is worthwhile to mention some of the important properties of the correlation.

**Proposition 3.1.** *If  $X$  and  $Y$  are random variables with non-zero variance, then:*

1.  $-1 \leq \rho(X, Y) \leq 1$ .
2. *If  $X$  and  $Y$  are independent, then  $\rho(X, Y) = 0$ .*
3.  $\rho(X, Y) = 1$  *if and only if  $X = aY + b$ , where  $a > 0, b \in \mathbb{R}$  (perfect correlation).*

4.  $\rho(X, Y) = -1$  if and only if  $X = aY + b$ , where  $a < 0, b \in \mathbb{R}$  (perfect anticorrelation).

Another important measure is the correlation ratio, it will be introduced before the definition of the possibilistic correlation ratio in Chapter 5.

### 3.2 Fuzzy sets, fuzzy numbers and possibility theory

Fuzzy sets were introduced by Zadeh in 1965 [106]. The concept of fuzzy sets extends the definition of the classical (crisp) sets. In the classical (bi-valued) case, the membership value of an element can take two possible values: 1 (it belongs to the set) or 0 (it does not belong to the set). In fuzzy logic this restriction is weakened by allowing any membership values from the unit interval. Formally, a fuzzy set,  $A$ , is a set of ordered pairs, where the first element is taken from a crisp set  $X$  and the second element is a value from  $[0, 1]$ . Based on this definition, we can construct a function:

$$\mu_A : X \rightarrow [0, 1],$$

which is termed as the membership function.

A *fuzzy number*  $A$  is a fuzzy set in  $\mathbb{R}$  with a normal (there exists an  $x \in \mathbb{R}$  such that  $\mu_A(x) = 1$ ), fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by  $\mathcal{F}$ . Fuzzy numbers can be considered as possibility distributions:

$$\text{Pos}(B \subset \mathbb{R}) = \sup_{x \in B} \mu_A(x)$$

A  $\gamma$ -level set of a fuzzy set  $A$  in  $\mathbb{R}^m$  is defined by  $[A]^\gamma = \{x \in \mathbb{R}^m : A(x) \geq \gamma\}$  if  $\gamma > 0$  and  $[A]^\gamma = \text{cl}\{x \in \mathbb{R}^m : A(x) > \gamma\}$  (the closure of the support of  $A$ ) if  $\gamma = 0$ . A fuzzy set  $C$  in  $\mathbb{R}^2$  is said to be a joint possibility distribution of fuzzy numbers  $A, B \in \mathcal{F}$ , if it satisfies the relationships

$$\max\{y \in \mathbb{R} \mid C(x, y)\} = A(x) \text{ and } \max\{x \in \mathbb{R} \mid C(x, y)\} = B(y),$$

for all  $x, y \in \mathbb{R}$ . Furthermore,  $A$  and  $B$  are called the marginal possibility distributions of  $C$ . In the following we will suppose that  $C$  is given in such a way that a uniform distribution can be defined on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ . Marginal possibility distributions are always uniquely defined from their joint possibility distribution by the *principle of falling shadows*.

Let  $C$  be a joint possibility distribution with marginal possibility distributions  $A, B \in \mathcal{F}$ , and let  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$  and  $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ ,  $\gamma \in [0, 1]$ . Then  $A$  and  $B$  are said to be non-interactive if their joint possibility distribution is  $A \times B$ ,

$$C(x, y) = \min\{A(x), B(y)\},$$

for all  $x, y \in \mathbb{R}$ , which can be written in the form,  $[C]^\gamma = [A]^\gamma \times [B]^\gamma$ , that is,  $[C]^\gamma$  is rectangular subset of  $\mathbb{R}^2$ , for any  $\gamma \in [0, 1]$ . If  $A$  and  $B$  are non-interactive then for any  $x \in [A]^\gamma$  and any  $y \in [B]^\gamma$  the ordered pair  $(x, y)$  will be in  $[C]^\gamma$  for any  $\gamma \in [0, 1]$ .

Another extreme situation is when  $[C]^\gamma$  is a line segment in  $\mathbb{R}^2$ . For example, let the *diagonal beam*,

$$C(x, y) = x\chi_{\{x=y\}}(x, y),$$

for any  $x, y \in [0, 1]$ , be the joint possibility distribution of marginal possibility distributions  $A(x) = x$  and  $B(y) = y$ . Then  $[C]^\gamma$  is a line segment  $[(\gamma, \gamma), (1, 1)]$  in  $\mathbb{R}^2$  for any  $\gamma \in [0, 1]$ . Furthermore, if one takes a point,  $x$ , from the  $\gamma$ -level set of  $A$  then one can take only  $y = x$  from the  $\gamma$ -level set of  $B$  for the pair  $(x, y)$  to belong to  $[C]^\gamma$ .

In possibility theory we can use the principle of *average value* of appropriately chosen real-valued functions to define mean value, variance, covariance and correlation of possibility distributions. A function  $f: [0, 1] \rightarrow \mathbb{R}$  is said to be a weighting function if  $f$  is non-negative, monotone increasing and satisfies the following normalization condition:

$$\int_0^1 f(\gamma) d\gamma = 1.$$

Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions.

The  $f$ -weighted possibilistic mean value of a possibility distribution  $A \in \mathcal{F}$  is the  $f$ -weighted average of probabilistic mean values of the respective uniform distributions on the level sets of  $A$  (recall that the mean value of  $U_\gamma$  is  $M(U_\gamma) = 1/2(a_1(\gamma) + a_2(\gamma))$  and its variance is computed by  $\text{var}(U_\gamma) = 1/12(a_2(\gamma) - a_1(\gamma))^2$ ).). That is, the  $f$ -weighted *possibilistic mean value* of  $A \in \mathcal{F}$ , with  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ,  $\gamma \in [0, 1]$ , is defined by [38],

$$E_f(A) = \int_0^1 M(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma, \quad (3.1)$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$  for all  $\gamma \in [0, 1]$ . This definition is based on the Goetschel-Voxman ordering of fuzzy numbers [41].

It should be noted that the choice of uniform probability distribution on the level sets of possibility distributions is not without reason. It is assumed that each point of a given level set is equally possible and then Laplace's principle of Insufficient Reason can be applied: if elementary events are equally possible, they should be equally probable (for more details and generalization of principle of Insufficient Reason see [24]). The idea of equipping the alpha-cuts with uniform probability is not new and refers to early ideas of simulation of fuzzy sets by Yager [102], and possibility/probability transforms by Dubois et al [30] as well as the pignistic transform of Smets [89].

For  $f(\gamma) \equiv 1$ , the  $f$ -weighted possibilistic mean value coincides with the (i) generative expectation of fuzzy numbers introduced by Chanas and M. Nowakowski in [14]; (ii) middle-point-of-the-mean-interval defuzzification method proposed by Yager [102].

There exist several other ways to define mean values of fuzzy numbers: Dubois and Prade [28] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. Using evaluation measures, Yoshida et al. [105] introduced a possibility mean, a necessity mean and a credibility mean of fuzzy numbers

that are different from (3.1). Surveying the results in quantitative possibility theory, Dubois [24] showed that some notions (e.g. cumulative distributions, mean values) in statistics can naturally be interpreted in the language of possibility theory.

The  $f$ -weighted possibilistic covariance between marginal possibility distributions of a joint possibility distribution is defined as the  $f$ -weighted average of probabilistic covariances between marginal probability distributions whose joint probability distribution is uniform on each level-set of the joint possibility distribution. That is, the  $f$ -weighted *possibilistic covariance* between  $A, B \in \mathcal{F}$ , (with respect to their joint distribution  $C$ ), can be written as [39],

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma. \quad (3.2)$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\text{cov}(X_\gamma, Y_\gamma)$  denotes their probabilistic covariance. It should be noted that the possibilistic covariance is an absolute measure of interactivity between marginal possibility distributions.

The measure of  $f$ -weighted possibilistic variance of  $A$  is the  $f$ -weighted average of the probabilistic variances of the respective uniform distributions on the level sets of  $A$ . That is, the  $f$ -weighted *possibilistic variance* of  $A$  is defined as the covariance of  $A$  with itself [39]

$$\text{Var}_f(A) = \text{Cov}_f(A, A) = \int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma. \quad (3.3)$$

There exist other approaches to define variance of fuzzy numbers: Dubois et al. [31] defined the potential variance of a symmetric fuzzy interval by viewing it as a family of its  $\alpha$ -cut.

There exist several other definitions of expected value, variance and covariance of *fuzzy random variables*, e.g. Kwakernaak [53, 54], Puri and Ralescu [82], Körner [55], Watanabe and Imaizumi [100], Feng et al. [35], Näther [78] and Shapiro [88].



## Chapter 4

# Possibilistic correlation of Fuzzy numbers

A measure of possibilistic covariance between marginal possibility distributions of a joint possibility distribution can be defined as the  $f$ -weighted average of probabilistic covariances between marginal probability distributions whose joint probability distribution is defined to be uniform on the  $\gamma$ -level sets of their joint possibility distribution [39]. This is an absolute measure of interactivity. A measure of possibilistic correlation between marginal possibility distributions of a joint possibility distribution is a relative measure of interactivity.

The main drawback of the measure of possibilistic correlation introduced in [13] that it does not necessarily take its values from  $[-1, 1]$  if some level-sets of the joint possibility distribution are not convex. A new normalization technique is needed.

In this chapter a new index of interactivity between marginal distributions of a joint possibility distribution will be introduced, which is defined for the whole family of joint possibility distributions. Namely, each level set of a joint possibility distribution is equipped with a uniform probability distribution, then the probabilistic correlation coefficient is computed between its marginal probability distributions, and then the new index of interactivity is defined as the weighted average of these coefficients over the set of all membership grades. These weights (or importances) can be given by weighting functions.

**Example 4.1.** Let  $A(x) = 1 - x$  and  $B(y) = 1 - y$  be fuzzy numbers with joint distribution,  $F$ , is defined by the Lukasiewicz  $t$ -norm. In this case

$$F(x, y) = \max\{A(x) + B(y) - 1, 0\} = \max\{1 - x - y, 0\},$$

and  $[F]^\gamma = \{(x, y) \mid x + y \leq 1 - \gamma\}$  is of symmetric triangular form for any  $0 \leq \gamma < 1$ . If we take, for example,  $\lambda = 0.4$  then the pair  $(0.3, 0.2)$  belongs to  $[F]^{0.4}$  since  $0.3 + 0.2 \leq 1 - 0.4$ , but the pair  $(0.4, 0.4)$  does not. In this approach a uniform probability distribution is defined on  $[F]^{0.4}$  with marginal probability distributions denoted by  $X_{0.4}$

and  $Y_{0.4}$ . The expected value of this uniform probability distribution,  $(0.2, 0.2)$ , will be nothing else but the center of mass (or gravity) of the set  $[F]^{0.4}$  of homogeneous density (for calculations see Section 4.2). Then the probabilistic correlation coefficient, denoted by  $\rho(X_{0.4}, Y_{0.4})$ , will be negative since the 'strength' of pairs  $(x, y) \in [F]^{0.4}$  that are discordant (i.e.  $(x - 0.2)(y - 0.2) < 0$ ) is bigger than the 'strength' of those ones that are concordant (i.e.  $(x - 0.2)(y - 0.2) > 0$ ). Then the index of interactivity is defined as the weighted average of these correlation coefficients over the set of all membership grades.

A measure of possibilistic correlation between marginal possibility distributions  $A$  and  $B$  of a joint possibility distribution  $C$  has been defined in [13] as their possibilistic covariance divided by the square root of the product of their possibilistic variances. That is, the  $f$ -weighted measure of *possibilistic correlation* of  $A, B \in \mathcal{F}$ , (with respect to their joint distribution  $C$ ), is defined as [13],

$$\begin{aligned} \rho_f^{\text{old}}(A, B) &= \frac{\text{Cov}_f(A, B)}{\sqrt{\text{Var}_f(A)}\sqrt{\text{Var}_f(B)}} \\ &= \frac{\int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma}{\left(\int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma\right)^{1/2} \left(\int_0^1 \text{var}(V_\gamma) f(\gamma) d\gamma\right)^{1/2}} \end{aligned} \quad (4.1)$$

where  $U_\gamma$  and  $V_\gamma$  are uniform probability distributions on  $[A]^\gamma$  and  $[B]^\gamma$ , respectively. Thus, the possibilistic correlation represents an average degree to which  $X_\gamma$  and  $Y_\gamma$  are linearly associated as compared to the dispersions of  $U_\gamma$  and  $V_\gamma$ . We have the following result [13]: if  $[C]^\gamma$  is convex for all  $\gamma \in [0, 1]$  then  $-1 \leq \rho_f^{\text{old}}(A, B) \leq 1$  for any  $f$ .

The presence of weighting function is not crucial in the theory: it can be simply removed from consideration by choosing  $f(\gamma) \equiv 1$ .

**Note 4.1.** *There exist several other ways to define correlation coefficient for fuzzy numbers, e.g. Liu and Kao [68] used fuzzy measures to define a fuzzy correlation coefficient of fuzzy numbers and they formulated a pair of nonlinear programs to find the  $\alpha$ -cut of this fuzzy correlation coefficient, then, in a special case, Hong [48] showed an exact calculation formula for this fuzzy correlation coefficient. Vaidyanathan [95] introduced a new measure for the correlation coefficient between triangular fuzzy variables called credibilistic correlation coefficient.*

## 4.1 An improved index of interactivity for fuzzy numbers

The main drawback of the definition of the former index of interactivity (4.1) is that it does not necessarily take its values from  $[-1, 1]$  if some level-sets of the joint possibility distribution are not convex. For example, consider a joint possibility distribution defined by

$$C(x, y) = 4x \cdot \chi_T(x, y) + 4/3(1 - x) \cdot \chi_S(x, y), \quad (4.2)$$



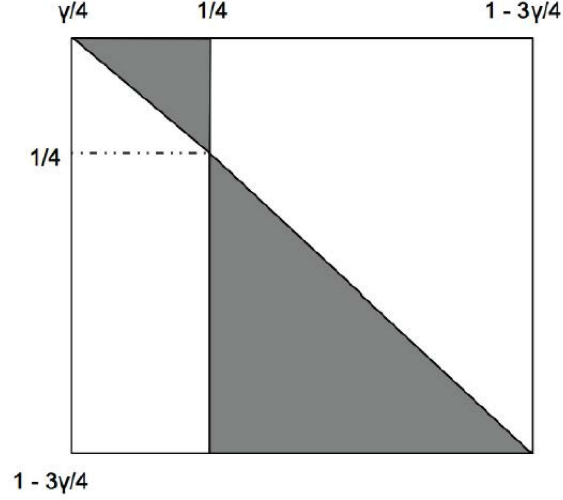


Figure 4.1: Not convex  $\gamma$ -level set.

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1/4, 0 \leq y \leq 1/4, x \leq y\},$$

$$S = \{(x, y) \in \mathbb{R}^2 \mid 1/4 \leq x \leq 1, 1/4 \leq y \leq 1, y \leq x\},$$

and

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid \gamma/4 \leq x \leq 1/4, x \leq y \leq \gamma/4\} \cup \{(x, y) \in \mathbb{R}^2 \mid 1/4 \leq x \leq 1 - 3/4\gamma, 1/4 \leq y \leq x\}.$$

It can be easily seen that  $[C]^\gamma$  is not a convex set for any  $\gamma \in [0, 1)$  (see Fig. 4.1).

Then the marginal possibility distributions of (4.2) are computed by

$$A(x) = B(x) = \begin{cases} 4x, & \text{if } 0 \leq x \leq 1/4 \\ \frac{4}{3}(1 - x), & \text{if } 1/4 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

After some computations one obtains that  $\rho_f^{old}(A, B) \approx 1.562$  for any weighting function  $f$ . The result is a value bigger than one since the variance of the first marginal distribution,  $X_\gamma$ , exceeds the variance of the uniform distribution on the same support.

Now a new index of interactivity is introduced between marginal distributions  $A$  and  $B$  of a joint possibility distribution  $C$  as the  $f$ -weighted average of the probabilistic correlation coefficients between the marginal probability distributions of a uniform probability distribution on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ . That is,

**Definition 4.1.** The  $f$ -weighted index of interactivity of  $A, B \in \mathcal{F}$  (with respect to their joint distribution  $C$ ) is defined by

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma \quad (4.3)$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}}$$

and, where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ .

In other words, the ( $f$ -weighted) index of interactivity is nothing else, but the  $f$ -weighted average of the probabilistic correlation coefficients  $\rho(X_\gamma, Y_\gamma)$  for all  $\gamma \in [0, 1]$ . It is clear that for any joint possibility distribution this new correlation coefficient always takes its value from the interval  $[-1, 1]$ , since  $\rho(X_\gamma, Y_\gamma) \in [-1, 1]$  for any  $\gamma \in [0, 1]$  and  $\int_0^1 f(\gamma) d\gamma = 1$ . As for the joint possibility distribution defined by (4.2) we get  $\rho_f(A, B) \approx 0.786$  for any  $f$ . Since  $\rho_f(A, B)$  measures an average index of interactivity between the level sets of  $A$  and  $B$ , sometimes this measure will be termed as the  $f$ -weighted possibilistic correlation coefficient.

## 4.2 An example

Consider the case, when  $A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by  $F(x, y) = (1 - x - y) \cdot \chi_T(x, y)$ , where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}.$$

A  $\gamma$ -level set of  $F$  is computed by

$$[F]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[F]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[F]^\gamma} dx dy}, & \text{if } (x, y) \in [F]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{(1 - \gamma)^2}, & \text{if } (x, y) \in [F]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(1 - \gamma - x)}{(1 - \gamma)^2}, & \text{if } 0 \leq x \leq 1 - \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{2(1-\gamma-y)}{(1-\gamma)^2}, & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

One can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[F]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} x(1-\gamma-x)dx = \frac{1-\gamma}{3}$$

and,

$$M(Y_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} y(1-\gamma-y)dy = \frac{1-\gamma}{3}.$$

The variations of  $X_\gamma$  and  $Y_\gamma$  can be calculated with the formula  $\text{var}(X) = M(X^2) - M(X)^2$ :

$$M(X_\gamma^2) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} x^2(1-\gamma-x)dx = \frac{(1-\gamma)^2}{6}$$

and,

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^2}{6} - \frac{(1-\gamma)^2}{9} = \frac{(1-\gamma)^2}{18}.$$

And similarly

$$\text{var}(Y_\gamma) = \frac{(1-\gamma)^2}{18}.$$

Using that

$$M(X_\gamma Y_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} \int_0^{1-\gamma-x} xy dy dx = \frac{(1-\gamma)^2}{12},$$

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(1-\gamma)^2}{36},$$

the probabilistic correlation of the random variables can be calculated as

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$ :

$$\rho_f(A, B) = \int_0^1 -\frac{1}{2}f(\gamma)d\gamma = -\frac{1}{2}.$$

Using the former definition (4.1) one would obtain  $\rho_f^{\text{old}}(A, B) = -1/3$  for the correlation coefficient (see [13] for details).

### 4.3 Some important cases

#### Non-interactive fuzzy numbers

If  $A$  and  $B$  are non-interactive then their joint possibility distribution is defined by  $C = A \times B$ . Since all  $[C]^\gamma$  are rectangular and the probability distribution on  $[C]^\gamma$  is defined to be uniform we get  $\text{cov}(X_\gamma, Y_\gamma) = 0$ , for all  $\gamma \in [0, 1]$ . So  $\text{Cov}_f(A, B) = 0$  and  $\rho_f(A, B) = 0$  for any weighting function  $f$ .

#### Perfect correlation

Fuzzy numbers  $A$  and  $B$  are said to be in perfect correlation, if there exist  $q, r \in \mathbb{R}$ ,  $q \neq 0$  such that their joint possibility distribution is defined by [13]

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1+r=x_2\}}(x_1, x_2) = B(x_2) \cdot \chi_{\{qx_1+r=x_2\}}(x_1, x_2), \quad (4.4)$$

where  $\chi_{\{qx_1+r=x_2\}}$ , stands for the characteristic function of the line

$$\{(x_1, x_2) \in \mathbb{R}^2 | qx_1 + r = x_2\}.$$

In this case

$$[C]^\gamma = \{(x, qx + r) \in \mathbb{R}^2 | x = (1-t)a_1(\gamma) + ta_2(\gamma), t \in [0, 1]\}$$

where  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ; and  $[B]^\gamma = q[A]^\gamma + r$ , for any  $\gamma \in [0, 1]$ , and, finally,

$$B(x) = A\left(\frac{x-r}{q}\right),$$

for all  $x \in \mathbb{R}$ . Furthermore,  $A$  and  $B$  are in a perfect positive [see Fig. 4.3] (negative [see Fig. 4.2]) correlation if  $q$  is positive (negative) in (4.4).

If  $A$  and  $B$  have a perfect positive (negative) correlation then from  $\rho(X_\gamma, Y_\gamma) = 1$  ( $\rho(X_\gamma, Y_\gamma) = -1$ ) [see [13] for details], for all  $\gamma \in [0, 1]$ , we get  $\rho_f(A, B) = 1$  ( $\rho_f(A, B) = -1$ ) for any weighting function  $f$ .

#### The case of pure shadows

Consider the case, when the joint possibility distribution is nothing else but the marginal distributions themselves. Let

$$A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, 1-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by 'pure shadows' (the marginal distributions themselves),

$$C(x, y) = (1-x-y) \cdot \chi_T(x, y),$$

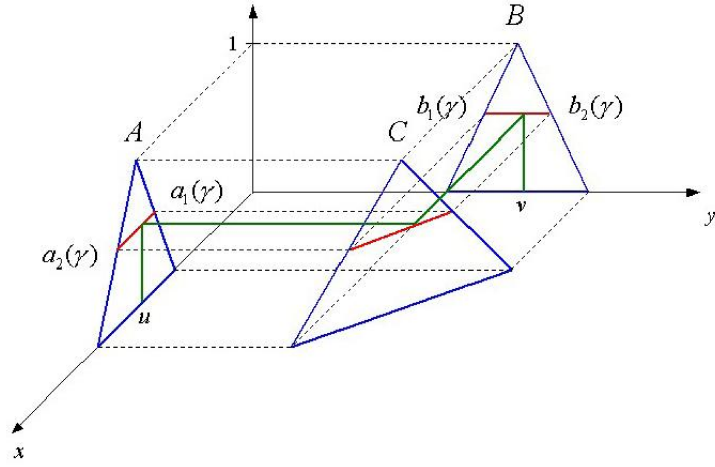


Figure 4.2: Perfect negative correlation.

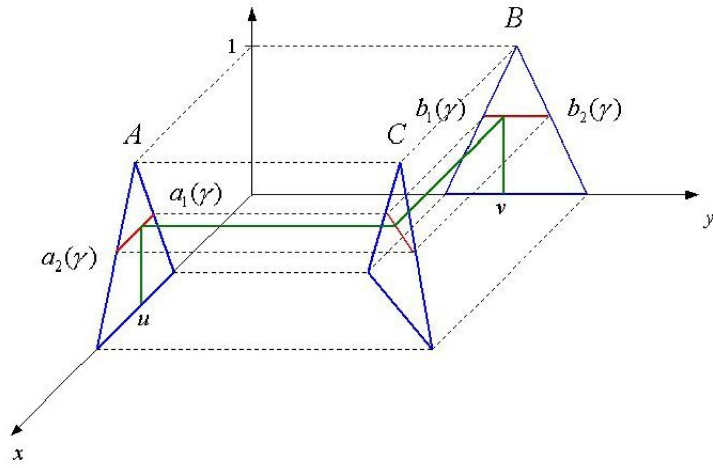


Figure 4.3: Perfect positive correlation.

where

$$T = \{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\} \cup \{(0, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}.$$

That is,

$$C(x, y) = \begin{cases} A(x) & \text{if } y = 0 \\ B(y) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 - \gamma\} \cup \{(0, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 - \gamma\}.$$

Since all  $\gamma$ -level sets of  $C$  are degenerated, i.e. their integrals vanish, everything can be computed as a limit of integrals. All the quantities are calculated with the  $\gamma$ -level sets:

$$[C]_\delta^\gamma = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 - \gamma, 0 \leq y \leq \delta\} \cup \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 - \gamma, 0 \leq x \leq \delta\}.$$

First the expected value and variance of  $X_\gamma$  and  $Y_\gamma$  is calculated as:

$$M(X_\gamma) = \lim_{\delta \rightarrow 0} \frac{1}{\int_{[C]_\delta^\gamma} dy dx} \int_{[C]_\delta^\gamma} x dx = \frac{1 - \gamma}{4},$$

$$M(X_\gamma^2) = \lim_{\delta \rightarrow 0} \frac{1}{\int_{[C]_\delta^\gamma} dy dx} \int_{[C]_\delta^\gamma} x^2 dx = \frac{(1 - \gamma)^2}{6},$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1 - \gamma)^2}{6} - \frac{(1 - \gamma)^2}{16} = \frac{5(1 - \gamma)^2}{48}.$$

Because of the symmetry, the results are the same for  $Y_\gamma$ . We need to calculate their covariance,

$$M(X_\gamma Y_\gamma) = \lim_{\delta \rightarrow 0} \frac{1}{\int_{[C]_\delta^\gamma} dx dy} \int_{[C]_\delta^\gamma} xy dy dx = 0,$$

Using this we obtain,

$$\text{cov}(X_\gamma, Y_\gamma) = -\frac{(1 - \gamma)^2}{16},$$

and for the correlation,

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}} = -\frac{3}{5}.$$

Finally we obtain the  $f$ -weighted possibilistic correlation:

$$\rho_f(A, B) = \int_0^1 -\frac{3}{5} f(\gamma) d\gamma = -\frac{3}{5}.$$

In this extremal case, the joint distribution is unequivocally constructed from the knowledge that  $C(x, y) = 0$  for any interior point  $(x, y)$  of the unit square. The reason for this negative correlation is the following: for example, for  $\gamma = 0.4$  the center of mass of  $[C]^{0.4}$  is  $(0.15, 0.15)$  and the crucial point here is that if any point,  $x$ , is chosen from  $[A]^{0.4}$  then the only possible choice from  $[B]^{0.4}$  can be  $y = 0$ , and this  $y = 0$  is always less than 0.15, independently of the choice of  $x$ . In  $[C]^{0.4}$  the strength of discordant points is much bigger than the strength of concordant points, with respect to the reference point  $(0.15, 0.15)$ .

**Joint Distribution:**  $(1 - \sqrt{x} - \sqrt{y})$

Consider the case, when

$$A(x) = B(x) = (1 - \sqrt{x}) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, (1 - \gamma)^2]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - \sqrt{x} - \sqrt{y}) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{6}{(1 - \gamma)^4}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{6(1 - \gamma - \sqrt{x})^2}{(1 - \gamma)^4}, & \text{if } 0 \leq x \leq (1 - \gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{6(1 - \gamma - \sqrt{y})^2}{(1 - \gamma)^4}, & \text{if } 0 \leq y \leq (1 - \gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

One can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$  :

$$M(X_\gamma) = \frac{6}{(1 - \gamma)^4} \int_0^{(1 - \gamma)^2} x(1 - \gamma - \sqrt{x})^2 dx = \frac{(1 - \gamma)^2}{5}$$

$$M(Y_\gamma) = \frac{6}{(1 - \gamma)^4} \int_0^{(1 - \gamma)^2} y(1 - \gamma - \sqrt{y})^2 dy = \frac{(1 - \gamma)^2}{5}$$

The variations of  $X_\gamma$  and  $Y_\gamma$  can be obtained as

$$M(X_\gamma^2) = \frac{6}{(1 - \gamma)^4} \int_0^{(1 - \gamma)^2} x^2(1 - \gamma - \sqrt{x})^2 dx = \frac{(1 - \gamma)^4}{14}$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^4}{14} - \frac{(1-\gamma)^4}{25} = \frac{9(1-\gamma)^4}{350}.$$

And similarly

$$\text{var}(Y_\gamma) = \frac{9(1-\gamma)^4}{350}.$$

Using that

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{13(1-\gamma)^4}{700},$$

the probabilistic correlation of the random variables is:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{13}{18} \approx -0.722.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$ :

$$\rho_f(A, B) = -\int_0^1 \frac{13}{18} f(\gamma) d\gamma = -\frac{13}{18}.$$

#### 4.4 Upper bound for the correlation in a special case

The purpose of this section is to calculate the possibilistic correlation of two fuzzy numbers,  $A(x)$  and  $B(x)$ , in the case when their joint distribution is

$$C(x, y) = \begin{cases} A(x), & \text{if } y = 0 \\ B(y), & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

From that it follows, that  $[C]^\gamma = [A]^\gamma \cup [B]^\gamma$ . First a lemma about probability distributions with the same structure is needed: a distribution, what is uniform on the set  $\{(x, 0) \mid a \leq x \leq b\} \cup \{(0, y) \mid a \leq y \leq b\}$ . The marginals are  $X$ , with the support  $[a, b]$ , and  $Y$ , with the support  $[c, d]$ . The notation  $Z$  will stand for the uniform joint distribution. The following lemma can be proven about the probabilistic correlation of  $X$  and  $Y$ :

**Lemma 4.1.**

$$\rho(X, Y) = -3\sqrt{\frac{t(a, b, c, d)}{s(a, b, c, d)}},$$

where  $t(a, b, c, d) = (b^2 - a^2)^2(d^2 - c^2)^2$  and  $s(a, b, c, d) = [4(b - a + d - c)(b^3 - a^3) - 3(b^2 - a^2)^2][4(b - a + d - c)(d^3 - c^3) - 3(d^2 - c^2)^2]$ .

*Proof.* It is easy to prove, that in this case

$$M(X) = \frac{b^2 - a^2}{2(b - a + d - c)}, \quad M(X^2) = \frac{b^3 - a^3}{3(b - a + d - c)},$$



$$M(Y) = \frac{d^2 - c^2}{2(b - a + d - c)}, \quad M(Y^2) = \frac{d^3 - c^3}{3(b - a + d - c)}.$$

From these results we obtain the variance of  $X$  and  $Y$ :

$$\text{var}(X) = M(X^2) - M(X)^2 = \frac{4(b - a + d - c)^2(b^3 - a^3) - 3(b^2 - a^2)^2}{12(b - a + d - c)^2},$$

$$\text{var} = M(Y^2) - M(Y)^2 = \frac{4(b - a + d - c)^2(d^3 - c^3) - 3(d^2 - c^2)^2}{12(b - a + d - c)^2}.$$

Since  $M(XY) = 0$ , the covariance has the following form:

$$\text{cov}(X, Y) = -\frac{(b^2 - a^2)(d^2 - c^2)}{4(b - a + d - c)^2}.$$

Now the correlation can be calculated easily:

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} = -3\sqrt{\frac{t(a, b, c, d)}{s(a, b, c, d)}}.$$

□

Since  $-1 \leq \rho(X, Y) \leq 1$  for any distributions, the following can be seen easily:

**Corrolary 4.1.** *For any  $a, b, c, d$  real numbers, such that  $a < b, c < d$ :*

$$\sqrt{\frac{t(a, b, c, d)}{s(a, b, c, d)}} \leq \frac{1}{3}.$$

The values  $a, b, c, d$  determine the value of the correlation, so the notation  $\rho(a, b, c, d)$  can be used.

**Example 4.2.** *Suppose  $a = c = 0$ , then the following results can be obtained:*

$$M(X) = \frac{b^2}{2(b + d)}, \quad M(X^2) = \frac{b^3}{3(b + d)}, \quad \text{var}(X) = \frac{b^3(b + 4d)}{12(b + d)^2},$$

$$M(Y) = \frac{d^2}{2(b + d)}, \quad M(Y^2) = \frac{d^3}{3(b + d)}, \quad \text{var}(Y) = \frac{d^3(d + 4b)}{12(b + d)^2},$$

$$\text{cov}(X, Y) = -\frac{b^2d^2}{4(b + d)^2},$$

$$\rho(X, Y) = -3\sqrt{\frac{bd}{(b + 4d)(d + 4b)}}.$$

*If  $b = d = 1$ , then  $\rho(X, Y) = -\frac{3}{5}$ .*

If  $a = c$  and  $b = d$ , the formula is simplified into the form:

$$\rho(X, Y) = -\frac{3(b+a)^2}{5(b+a)^2 - 8ab}.$$

From the example one can see, that this is  $-\frac{3}{5}$ , in the case when  $a = 0$  and  $b = 1$ . It is necessary now to show, that this is the upper bound, if  $0 \leq a$ .

$$\rho(X, Y) = -\frac{3(b+a)^2}{5(b+a)^2 - 8ab} \leq -\frac{3}{5}$$

holds if and only if

$$\frac{(b+a)^2}{5(b+a)^2 - 8ab} \geq -\frac{1}{5} \Leftrightarrow 5(b+a)^2 \geq 5(b+a)^2 - 8ab,$$

which is true if and only if  $0 \geq -8ab$ , so from  $0 \leq a \leq b$  it can be seen that the correlation in this case is always between  $-1$  and  $-\frac{3}{5}$ . And in the example distributions were found with correlation coefficient  $-\frac{3}{5}$ , so this is the least upper bound.

Any fuzzy number can be described in the following form [27]:

$$A(x) = \begin{cases} L(\frac{a-x}{\alpha}), & \text{if } a-\alpha \leq x \leq a \\ 1, & \text{if } a \leq x \leq b \\ R(\frac{x-b}{\beta}), & \text{if } b \leq x \leq b+\beta \\ 0 & \text{otherwise} \end{cases}$$

where  $[a, b]$  is the peak of  $A$ ,  $a$  and  $b$  are the lower and upper modal values;  $L$  and  $R$  are shape functions:  $[0, 1] \rightarrow [0, 1]$ , with  $L(0) = R(0) = 1$  and  $L(1) = R(1) = 0$ , which are non-increasing, continuous mappings.

The previous observations about probability distributions can be used to describe the correlation in the mentioned case. In this case  $[C]^\gamma = [a_1(\gamma), a_2(\gamma)] \cup [b_1(\gamma), b_2(\gamma)]$ , with a uniform distribution on this set with marginals  $X_\gamma$  and  $Y_\gamma$ . Their correlation is exactly what was calculated before, with  $a = a_1(\gamma), b = a_2(\gamma), c = b_1(\gamma), d = b_2(\gamma)$ . It means that the possibilistic correlation can be written in the following form:

$$\rho_f(A, B) = \int_0^1 \rho(a_1(\gamma), a_2(\gamma), b_1(\gamma), b_2(\gamma)) f(\gamma) d\gamma \quad (4.5)$$

Since the numbers  $a_1(\gamma), a_2(\gamma), b_1(\gamma), b_2(\gamma)$  satisfy the requirements of the previous reasoning and  $f$  is a weighting function, it can be easily seen that in this case

$$\rho_f(A, B) \leq -\frac{3}{5}.$$

## 4.5 Portfolio optimization using the Possibilistic Correlation

In this section the use of the possibilistic correlation is demonstrated in portfolio optimization [70]. In this example there are 3 assets considered, which are represented by fuzzy numbers:

$$A_1(x) = A_2(x) = (1 - x^2) \cdot \chi_{[0,1]}(x)$$

and

$$A_3(x) = (1 - x) \cdot \chi_{[0,1]}(x).$$

It is easy to see that  $E(A_3) = \frac{1}{6}$ ,  $E(A_1) = E(A_2) = \frac{4}{15}$  and  $\text{Var}(A_3) = \frac{1}{72}$ ,  $\text{Var}(A_1) = \text{Var}(A_2) = \frac{1}{36}$ . In the following the pairwise correlation coefficients will be computed.

**Joint Distribution of  $(A_1, A_3)$  and  $(A_2, A_3)$ :**  $(1 - x^2 - y)$

The same joint distribution will be used for  $(A_1, A_3)$  and  $(A_2, A_3)$ , so it is sufficient to calculate the first one. Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - x^2 - y) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{2(1 - \gamma)^{\frac{3}{2}}}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_{A_1}(x) = \begin{cases} \frac{3(1 - \gamma - x^2)}{2(1 - \gamma)^{\frac{3}{2}}}, & \text{if } 0 \leq x \leq \sqrt{1 - \gamma} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{A_3}(y) = \begin{cases} \frac{3\sqrt{1 - \gamma - y}}{2(1 - \gamma)^{\frac{3}{2}}}, & \text{if } 0 \leq y \leq 1 - \gamma \\ 0 & \text{otherwise} \end{cases}$$

One can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$  :

$$M(X_\gamma) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{\sqrt{1-\gamma}} x(1-\gamma-x^2)dx = \frac{3\sqrt{1-\gamma}}{8}$$

$$M(Y_\gamma) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{\sqrt{1-\gamma}} y\sqrt{1-\gamma-y}dy = \frac{2(1-\gamma)}{5}.$$

The variations of  $X_\gamma$  and  $Y_\gamma$  can be obtained as:

$$M(X_\gamma^2) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{\sqrt{1-\gamma}} x^2(1-\gamma-x^2)dx = \frac{1-\gamma}{5}$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{1-\gamma}{5} - \frac{9(1-\gamma)}{64} = \frac{19(1-\gamma)}{320}.$$

$$M(Y_\gamma^2) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{1-\gamma} y^2\sqrt{1-\gamma-y}dy = \frac{8(1-\gamma)^2}{35}.$$

$$\text{var}(Y_\gamma) = M(Y_\gamma^2) - M(Y_\gamma)^2 = \frac{8(1-\gamma)^2}{35} - \frac{4(1-\gamma)^2}{25} = \frac{12(1-\gamma)^2}{175}.$$

The covariance of  $X_\gamma$  and  $Y_\gamma$ :

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(1-\gamma)^{\frac{3}{2}}}{40},$$

and one can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\sqrt{\frac{35}{228}} \approx -0.392.$$

And finally the  $f$ -weighted possibilistic correlation of  $A_1$  and  $A_3$  (and also of  $A_2$  and  $A_3$ ):

$$\rho_f(A_1, A_3) = \int_0^1 -\sqrt{\frac{35}{228}}f(\gamma)d\gamma = -\sqrt{\frac{35}{228}}.$$

**Joint distribution of  $(A_1, A_2)$ :**  $(1-x^2-y^2)$

Suppose that the joint possibility distribution of  $A_1$  and  $A_2$  is given by:

$$C(x, y) = (1-x^2-y^2) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{4}{(1 - \gamma)\pi}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_{A_1}(x) = \begin{cases} \frac{4\sqrt{1 - \gamma - x^2}}{(1 - \gamma)\pi}, & \text{if } 0 \leq x \leq 1 - \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$f_{A_2}(y) = \begin{cases} \frac{4\sqrt{1 - \gamma - y^2}}{(1 - \gamma)\pi}, & \text{if } 0 \leq y \leq 1 - \gamma \\ 0 & \text{otherwise} \end{cases}$$

One can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$  :

$$M(X_\gamma) = \frac{4}{(1 - \gamma)\pi} \int_0^{\sqrt{1 - \gamma}} x \sqrt{1 - \gamma - x^2} dx = \frac{4\sqrt{1 - \gamma}}{3\pi}$$

$$M(Y_\gamma) = \frac{4}{(1 - \gamma)\pi} \int_0^{\sqrt{1 - \gamma}} y \sqrt{1 - \gamma - y^2} dy = \frac{4\sqrt{1 - \gamma}}{3\pi}.$$

The variations of  $X_\gamma$  and  $Y_\gamma$  are:

$$M(X_\gamma^2) = \frac{4}{(1 - \gamma)\pi} \int_0^{\sqrt{1 - \gamma}} x^2 \sqrt{1 - \gamma - x^2} dx = \frac{1 - \gamma}{4}$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{1 - \gamma}{4} - \frac{16(1 - \gamma)}{9\pi^2} = \frac{(1 - \gamma)(9\pi^2 - 64)}{36\pi^2}.$$

And similarly

$$\text{var}(Y_\gamma) = \frac{(1 - \gamma)(9\pi^2 - 64)}{36\pi^2}.$$

Using that

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = \frac{(1 - \gamma)(9\pi - 32)}{18\pi^2},$$

the probabilistic correlation of the random variables can be obtained as

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = \frac{2(9\pi - 32)}{(9\pi^2 - 64)} \approx -0.302.$$

And finally the  $f$ -weighted possibilistic correlation of  $A_1$  and  $A_2$ :

$$\rho_f(A, B) = \int_0^1 \frac{2(9\pi - 32)}{(9\pi^2 - 64)} f(\gamma) d\gamma = \frac{2(9\pi - 32)}{(9\pi^2 - 64)}.$$

## Portfolio optimization

The traditional portfolio optimization problem has the following form:

$$\begin{aligned} \max \quad & \sum_{i=1}^n x_i E(A_i) - \\ & \theta \left( \sum_{i=1}^n x_i^2 \text{Var}(A_i) + \sum_{i=1}^n \sum_{j \neq i} x_i x_j \sqrt{\text{Var}(A_i) \text{Var}(A_j)} \rho(A_i, A_j) \right) \\ \text{subject to} \quad & \sum_{i=1}^n x_i = 1 \end{aligned} \quad (4.6)$$

where  $x_i$  is the weight of asset  $A_i$  and  $\theta$  is the risk tolerance factor.

After solving this simple quadratic optimization problem with the parameters obtained before, one can find that for all  $\theta < 2,88$ , the optimal portfolio is obtained when the money is equally distributed between  $A_1$  and  $A_2$ , and  $A_3$  is not included in the portfolio ( $x_1 = x_2 = \frac{1}{2}, x_3 = 0$ .) If  $\theta$  increases (which means that assets with large variance are penalized),  $x_3$  starts to increase, since the added variance of  $A_3$  to the portfolio is low, and for large values of  $\theta$ ,  $A_3$  becomes the dominant asset of the portfolio.

## 4.6 Trapezoidal marginal distributions

In this section the correlation of two trapezoidal fuzzy numbers will be calculated. Consider now the case,

$$A(x) = B(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } 1 \leq x \leq 2 \\ 3 - x, & \text{if } 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [\gamma, 3 - \gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that the joint possibility distribution of these two trapezoidal marginal distributions - a truncated pyramid - is given by:

$$C(x, y) = \begin{cases} y, & \text{if } 0 \leq x \leq 3, 0 \leq y \leq 1, x \leq y, x \leq 3 - y \\ 1, & \text{if } 1 \leq x \leq 2, 1 \leq y \leq 2, x \leq y \\ x, & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 3, y \leq x, x \leq 3 - y \\ 0, & \text{otherwise} \end{cases}$$

Then  $[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid \gamma \leq x \leq 3 - \gamma, \gamma \leq y \leq 3 - x\}$ .

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{(3 - 2\gamma)^2}, & \text{if } (x, y) \in [C]^\gamma \\ 0, & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(3-\gamma-x)}{(3-2\gamma)^2}, & \text{if } \gamma \leq x \leq 3-\gamma \\ 0 & \text{otherwise} \end{cases}$$

and,

$$f_2(y) = \begin{cases} \frac{2(3-\gamma-y)}{(3-2\gamma)^2}, & \text{if } \gamma \leq y \leq 3-\gamma \\ 0 & \text{otherwise} \end{cases}$$

One can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} x(3-\gamma-x)dx = \frac{\gamma+3}{3}$$

and,

$$M(Y_\gamma) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} y(3-\gamma-y)dy = \frac{\gamma+3}{3}$$

The variations of  $X_\gamma$  and  $Y_\gamma$  can be calculated as:

$$M(X_\gamma^2) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} x^2(3-\gamma-x)dx = \frac{2\gamma^2+9}{6}$$

and,

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{2\gamma^2+9}{6} - \frac{(\gamma+3)^2}{9} = \frac{(3-2\gamma)^2}{18}.$$

And similarly we obtain

$$\text{var}(Y_\gamma) = \frac{(3-2\gamma)^2}{18}.$$

Using the relationship,

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(3-2\gamma)^2}{36},$$

the probabilistic correlation of the random variables is obtained as:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$  is,

$$\rho_f(A, B) = -\int_0^1 \frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

## 4.7 Time Series With Fuzzy Data

A time series with fuzzy data is referred to as fuzzy time series. Consider a fuzzy time series indexed by  $t \in (0, 1]$ ,

$$A_t(x) = \begin{cases} 1 - \frac{x}{t}, & \text{if } 0 \leq x \leq t \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad A_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that in this case,

$$[A_t]^\gamma = [0, t(1 - \gamma)], \quad \gamma \in [0, 1].$$

If  $t_1, t_2 \in [0, 1]$ , then the joint possibility distribution of the corresponding fuzzy numbers is given by:

$$C(x, y) = \left(1 - \frac{x}{t_1} - \frac{y}{t_2}\right) \cdot \chi_T(x, y),$$

where

$$T = \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \frac{x}{t_1} + \frac{y}{t_2} \leq 1 \right\}.$$

$$\text{Then } [C]^\gamma = \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \frac{x}{t_1} + \frac{y}{t_2} \leq 1 - \gamma \right\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

That is,

$$f(x, y) = \begin{cases} \frac{2}{t_1 t_2 (1 - \gamma)^2}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(1 - \gamma - \frac{x}{t_1})}{t_1(1 - \gamma)^2}, & \text{if } 0 \leq x \leq t_1(1 - \gamma) \\ 0 & \text{otherwise} \end{cases}$$

and,

$$f_2(y) = \begin{cases} \frac{2(1 - \gamma - \frac{y}{t_2})}{t_2(1 - \gamma)^2}, & \text{if } 0 \leq y \leq t_2(1 - \gamma) \\ 0 & \text{otherwise} \end{cases}$$

One can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$  :

$$M(X_\gamma) = \frac{2}{t_1(1 - \gamma)^2} \int_0^{t_1(1 - \gamma)} x(1 - \gamma - \frac{x}{t_1}) dx = \frac{t_1(1 - \gamma)}{3}$$



and

$$M(Y_\gamma) = \frac{2}{t_2(1-\gamma)^2} \int_0^{t_2(1-\gamma)} y(1-\gamma - \frac{y}{t_2}) dy = \frac{t_2(1-\gamma)}{3}.$$

We calculate now the variations of  $X_\gamma$  and  $Y_\gamma$  as,

$$M(X_\gamma^2) = \frac{2}{t_1(1-\gamma)^2} \int_0^{t_1(1-\gamma)} x^2(1-\gamma - \frac{x}{t_1}) dx = \frac{t_1^2(1-\gamma)^2}{6}$$

and,

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{t_1^2(1-\gamma)^2}{6} - \frac{t_1^2(1-\gamma)^2}{9} = \frac{t_1^2(1-\gamma)^2}{18}.$$

And in a similar way,

$$\text{var}(Y_\gamma) = \frac{t_2^2(1-\gamma)^2}{18}.$$

From,

$$\text{cov}(X_\gamma, Y_\gamma) = -\frac{t_1 t_2 (1-\gamma)^2}{36},$$

the probabilistic correlation of the random variables can be calculated as

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

The  $f$ -weighted possibilistic correlation of  $A_{t_1}$  and  $A_{t_2}$  is

$$\rho_f(A_{t_1}, A_{t_2}) = \int_0^1 -\frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

Thus the autocorrelation function of this fuzzy time series is constant. Namely,

$$R(t_1, t_2) = -\frac{1}{2},$$

for all  $t_1, t_2 \in [0, 1]$ .

## 4.8 Discussion

A novel measure of (relative) index of interactivity was introduced between marginal distributions  $A$  and  $B$  of a joint possibility distribution  $C$ . The starting point of the approach is to equip the  $\gamma$ -level sets of the joint possibility distribution with uniform probability distributions. Then the correlation coefficient between its marginal probability distributions is considered to be an index of interactivity between the  $\gamma$ -level sets of  $A$  and  $B$ . This new index of interactivity is meaningful for any joint possibility distribution and possesses similar properties to the probabilistic correlation coefficient.



## Chapter 5

# Possibilistic correlation ratio

In statistics, the correlation ratio is a measure of the relationship between the statistical dispersion within individual categories and the dispersion across the whole population or sample. The correlation ratio was originally introduced by Pearson [79] as part of analysis of variance and it was extended to random variables by Kolmogorov [51] as,

$$\eta^2(X|Y) = \frac{D^2[E(X|Y)]}{D^2(X)},$$

where  $X$  and  $Y$  are random variables. If  $X$  and  $Y$  have a joint probability density function, denoted by  $f(x, y)$ , then  $\eta^2(X|Y)$  can be computed using the following formulas:

$$E(X|Y = y) = \int_{-\infty}^{\infty} xf(x|y)dx$$

and

$$D^2[E(X|Y)] = E(E(X|y) - E(X))^2,$$

where

$$f(x|y) = \frac{f(x, y)}{f(y)}.$$

In recent years the correlation ratio is recognized as a key notion in global sensitivity analysis [72].

**Note 5.1.** *The correlation ratio measures the functional dependence between  $X$  and  $Y$ . It takes on values between 0 (no functional dependence) and 1 (purely deterministic dependence). It is worth noting that if  $E(X|Y = y)$  is a linear function of  $y$  (i.e. there is a linear relationship between random variables  $E(X|Y)$  and  $Y$ ) this will give the same result as the square of the correlation coefficient, otherwise the correlation ratio will be larger in magnitude. It can therefore be used for judging non-linear relationships. Also note that the correlation ratio is asymmetrical by nature since the two random*

variables fundamentally do not play the same role in the functional relationship; in general,  $\eta^2(X|Y) \neq \eta^2(Y|X)$ . One can obtain a symmetrical definition using

$$\eta^2(X, Y) = \max \{ \eta^2(X|Y), \eta^2(Y|X) \}.$$

Also important to note that the correlation ratio is invariant to multiplicative changes in the first argument:

$$\eta^2(kX|Y) = \eta^2(X|Y)$$

The following important theorem characterizes the relationship between the correlation and correlation ratio:

**Theorem 5.1.**

$$\eta^2(X|Y) = \sup_f \rho^2(X, f(Y)),$$

where the supremum is taken for all the functions  $f$ , such that  $f(Y)$  has finite variance. The correlation can reach its maximum if  $f(y) = aE(X|Y = y) + b$ .

The difference between  $\eta^2(X|Y)$  and  $\rho^2(X, Y)$  can be interpreted as the degree of non-linearity between  $X$  and  $Y$ :

$$\eta^2(X|Y) - \rho^2(X, Y) = \frac{1}{D^2(X)} \left\{ \min_{a,b} E(Y - (aX + b))^2 - \min_f E(Y - f(X))^2 \right\}.$$

**Example 5.1.** The use of the correlation ratio can be illustrated in a very simple example. Suppose we have two probability distributions,  $X$  and  $Y$ , with two-dimensional standard normal joint distribution, and the correlation coefficient of  $X$  and  $Y$  is  $r$ . Then the relationship between  $X$  and  $Y^2$  is clearly not linear, so their correlation coefficient is 0. But if  $r$  is close to 1, the relationship between  $X$  and  $Y^2$  is still very strong. And in this case the correlation ratio takes the value  $r^2$ .

## 5.1 A Correlation Ratio for Marginal Possibility Distributions

The principle of *expected value* of appropriately chosen functions on fuzzy sets can be used to define the correlation ratio of fuzzy numbers. Namely, one can equip each level set of a possibility distribution (represented by a fuzzy number) with a uniform probability distribution, then apply their standard probabilistic calculation, and then define measures on possibility distributions by integrating these weighted probabilistic notions over the set of all membership grades.

**Definition 5.1.** Let  $A$  and  $B$  be the marginal possibility distributions of a given joint possibility distribution  $C$ . Then the  $f$ -weighted possibilistic correlation ratio of marginal possibility distribution  $A$  with respect to marginal possibility distribution  $B$  is defined by

$$\eta_f^2(A|B) = \int_0^1 \eta^2(X_\gamma|Y_\gamma) f(\gamma) d\gamma \quad (5.1)$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\eta^2(X_\gamma|Y_\gamma)$  denotes their probabilistic correlation ratio.

The  $f$ -weighted possibilistic correlation ratio of the fuzzy number  $A$  on  $B$  is nothing else, but the  $f$ -weighted average of the probabilistic correlation ratios  $\eta^2(X_\gamma|Y_\gamma)$  for all  $\gamma \in [0, 1]$ .

The following properties can be easily checked based on the definition.

**Lemma 5.1.**     • *The correlation ratio takes on values between 0 and 1.*

- *The correlation ratio is asymmetrical since the marginal distributions fundamentally do not play the same role in the functional relationship; in general,  $\eta^2(A|B) \neq \eta^2(B|A)$ . A symmetrical definition can be obtained using*

$$\eta^2(A, B) = \max \{ \eta^2(A|B), \eta^2(B|A) \}.$$

- *If  $E(X_\gamma|Y_\gamma = y)$  is linear function of  $y$  (i.e. there is a linear relationship between random variables  $E(X_\gamma|Y_\gamma)$  and  $Y_\gamma$ ) for every  $\gamma \in [0, 1]$  this will give the same result as the square of the correlation coefficient ( $\rho(A, B)$ ), otherwise the correlation ratio will be larger in magnitude.*
- *If  $A$  and  $B$  are symmetrical fuzzy numbers, then*

$$\eta^2(A|B) = \eta^2(B|A) = 0.$$

## 5.2 Computation of Correlation Ratio: Some Examples

In this section the  $f$ -weighted possibilistic correlation ratio will be computed for joint possibility distributions  $(1 - x - y)$ ,  $(1 - x^2 - y)$ ,  $(1 - \sqrt{x} - y)$ ,  $(1 - x^2 - y^2)$  and  $(1 - \sqrt{x} - \sqrt{y})$  defined on proper subsets of the unit square.

### A Linear Relationship

Consider the case, when

$$A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by  $C(x, y) = (1 - x - y) \cdot \chi_T(x, y)$ , where

$$T = \{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1 \}.$$

Then  $[C]^\gamma = \{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1 - \gamma \}$ . The density function of a uniform distribution on  $[C]^\gamma$  is

$$f(x, y) = \begin{cases} \frac{2}{(1 - \gamma)^2} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(1-\gamma-x)}{(1-\gamma)^2} & \text{if } 0 \leq x \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{2(1-\gamma-y)}{(1-\gamma)^2} & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ration one needs to calculate the conditional probability distribution:

$$E(X|Y = y) = \int_0^{1-\gamma-y} xf(x|y)dx = \int_0^{1-\gamma-y} x \frac{f(x, y)}{f_2(y)} dx = \frac{1-\gamma-y}{2},$$

where  $0 \leq x \leq 1-\gamma$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(X|Y)] &= E(E(X|y) - E(X))^2 \\ &= \int_0^{1-\gamma} \left( \frac{1-\gamma-y}{2} - \frac{1-\gamma}{3} \right)^2 \frac{2(1-\gamma-y)}{(1-\gamma)^2} dy \\ &= \frac{(1-\gamma)^2}{72}. \end{aligned}$$

Using the relationship

$$D^2(X_\gamma) = \frac{(1-\gamma)^2}{18},$$

the probabilistic correlation of  $X_\gamma$  on  $Y_\gamma$  is obtained as

$$\eta^2(X_\gamma|Y_\gamma) = \frac{1}{4}.$$

From this the  $f$ -weighted possibilistic correlation ratio of  $A$  with respect to  $B$  is,

$$\eta_f^2(A|B) = \int_0^1 \frac{1}{4} f(\gamma) d\gamma = \frac{1}{4}.$$

**Note 5.2.** *In this simple case*

$$\eta_f^2(A|B) = \eta_f^2(B|A) = [\rho_f(A, B)]^2,$$

since  $E(X_\gamma|Y_\gamma = y)$  is a linear function of  $y$ :

$$\begin{aligned} E(X_\gamma|Y_\gamma = y) &= \frac{1-\gamma-y}{2} = \frac{1-\gamma}{3} - \frac{y}{2} + \frac{1-\gamma}{6} \\ &= \frac{1-\gamma}{3} - \frac{1}{2}y - \left( -\frac{1}{2} \right) \times \frac{1-\gamma}{3} \\ &= \frac{1-\gamma}{3} - \frac{1}{2} \left( y - \frac{1-\gamma}{3} \right) = E(X_\gamma) - \rho(X_\gamma, Y_\gamma)(y - E(Y_\gamma)). \end{aligned}$$

## A Nonlinear Relationship

Consider the case, when

$$A(x) = (1 - x^2) \cdot \chi_{[0,1]}(x),$$

$$B(x) = (1 - y) \cdot \chi_{[0,1]}(y),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [0, \sqrt{1-\gamma}]$ ,  $[B]^\gamma = [0, 1-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - x^2 - y) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{2(1-\gamma)^{\frac{3}{2}}} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{3(1-\gamma-x^2)}{2(1-\gamma)^{\frac{3}{2}}} & \text{if } 0 \leq x \leq \sqrt{1-\gamma} \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{3\sqrt{1-\gamma-y}}{2(1-\gamma)^{\frac{3}{2}}} & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ratio one needs to calculate the conditional probability distribution:

$$E(Y|X = x) = \int_0^{1-\gamma-x^2} y f(y|x) dy = \int_0^{1-\gamma-x^2} y \frac{f(x, y)}{f_1(x)} dy = \frac{1-\gamma-x^2}{2},$$

where  $0 \leq y \leq 1-\gamma$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{\sqrt{1-\gamma}} \left( \frac{1-\gamma-x^2}{2} - \frac{2(1-\gamma)}{5} \right)^2 \frac{3(1-\gamma-x^2)}{2(1-\gamma)^{\frac{3}{2}}} dx \\ &= \frac{2(1-\gamma)^2}{175}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{12(1-\gamma)^2}{175},$$

the probabilistic correlation ratio of  $Y_\gamma$  with respect to  $X_\gamma$  is obtained as

$$\eta^2(Y_\gamma|X_\gamma) = \frac{1}{6}.$$

From this the  $f$ -weighted possibilistic correlation ratio of  $B$  with respect to  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{1}{6} f(\gamma) d\gamma = \frac{1}{6}.$$

Similarly, from  $D^2[E(X|Y)] = \frac{3(1-\gamma)}{320}$ , and from

$$D^2(X_\gamma) = \frac{19(1-\gamma)}{320},$$

one obtains,

$$\eta_f^2(A|B) = \int_0^1 \frac{3}{19} f(\gamma) d\gamma = \frac{3}{19}.$$

That is  $\eta_f^2(B|A) \neq \eta_f^2(A|B)$ .

### **Joint Distribution:** $(1 - \sqrt{x} - y)$

Consider the case, when

$$A(x) = (1 - \sqrt{x}) \cdot \chi_{[0,1]}(x),$$

$$B(x) = (1 - y) \cdot \chi_{[0,1]}(y),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [0, (1-\gamma)^2]$ ,  $[B]^\gamma = [0, 1-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - \sqrt{x} - y) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + y \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + y \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{(1-\gamma)^3} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$



The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{3(1-\gamma-\sqrt{x})}{(1-\gamma)^3} & \text{if } 0 \leq x \leq (1-\gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{3(1-\gamma-y)^2}{(1-\gamma)^3} & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ratio we need to calculate the conditional probability distribution:

$$E(Y|X=x) = \int_0^{1-\gamma-\sqrt{x}} y f(y|x) dy = \int_0^{1-\gamma-\sqrt{x}} y \frac{f(x,y)}{f_1(x)} dy = \frac{1-\gamma-\sqrt{x}}{2},$$

where  $0 \leq y \leq 1-\gamma$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{(1-\gamma)^2} \left( \frac{1-\gamma-\sqrt{x}}{2} - \frac{1-\gamma}{4} \right)^2 \frac{3(1-\gamma-\sqrt{x})}{(1-\gamma)^3} dx \\ &= \frac{(1-\gamma)^2}{80}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{3(1-\gamma)^2}{80},$$

one obtains that the probabilistic correlation ratio of  $Y_\gamma$  with respect to  $X_\gamma$  is

$$\eta^2(Y_\gamma|X_\gamma) = \frac{1}{3}.$$

From this the  $f$ -weighted possibilistic correlation ratio of  $B$  with respect to  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{1}{3} f(\gamma) d\gamma = \frac{1}{3}.$$

Similarly, from  $D^2[E(X|Y)] = \frac{3(1-\gamma)^4}{175}$ , and from

$$D^2(X_\gamma) = \frac{37(1-\gamma)^4}{700},$$

we obtain:

$$\eta_f^2(A|B) = \int_0^1 \frac{12}{37} f(\gamma) d\gamma = \frac{12}{37}.$$

## A Ball-Shaped Joint Distribution

Consider the case, when

$$A(x) = B(x) = (1 - x^2) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, \sqrt{1-\gamma}]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is ball-shaped, that is,

$$C(x, y) = (1 - x^2 - y^2) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{4}{(1-\gamma)\pi} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{4\sqrt{1-\gamma-x^2}}{(1-\gamma)\pi} & \text{if } 0 \leq x \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{4\sqrt{1-\gamma-y^2}}{(1-\gamma)\pi} & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ratio, the conditional probability distribution has to be calculated:

$$E(Y|X=x) = \int_0^{\sqrt{1-\gamma-x^2}} y f(y|x) dy = \int_0^{\sqrt{1-\gamma-x^2}} y \frac{f(x,y)}{f_1(x)} dy = \frac{\sqrt{1-\gamma-x^2}}{2},$$

where  $0 \leq y \leq \sqrt{1-\gamma}$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{\sqrt{1-\gamma}} \left( \frac{\sqrt{1-\gamma-x^2}}{2} - \frac{4\sqrt{1-\gamma}}{3\pi} \right)^2 \frac{4\sqrt{1-\gamma-x^2}}{\pi(1-\gamma)} dx \\ &= \frac{(1-\gamma)(27\pi^2 - 256)}{144\pi^2}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{(1 - \gamma)(9\pi^2 - 64)}{36\pi^2},$$

one obtains that the probabilistic correlation ratio of  $Y_\gamma$  with respect to  $X_\gamma$  is

$$\eta^2(Y_\gamma|X_\gamma) = \frac{27\pi^2 - 256}{36\pi^2 - 256}.$$

Finally, we get that the  $f$ -weighted possibilistic correlation ratio of  $B$  with respect to  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{27\pi^2 - 256}{36\pi^2 - 256} f(\gamma) d\gamma = \frac{27\pi^2 - 256}{36\pi^2 - 256}.$$

**Joint Distribution:**  $(1 - \sqrt{x} - \sqrt{y})$

Consider the case, when

$$A(x) = B(x) = (1 - \sqrt{x}) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, (1 - \gamma)^2]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - \sqrt{x} - \sqrt{y}) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{6}{(1 - \gamma)^4} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{6(1 - \gamma - \sqrt{x})^2}{(1 - \gamma)^4} & \text{if } 0 \leq x \leq (1 - \gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{6(1 - \gamma - \sqrt{y})^2}{(1 - \gamma)^4} & \text{if } 0 \leq y \leq (1 - \gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ratio, the conditional probability distribution has to be computed:

$$E(Y|X = x) = \int_0^{(1-\gamma-\sqrt{x})^2} y f(y|x) dy = \int_0^{(1-\gamma-\sqrt{x})^2} y \frac{f(x, y)}{f_1(x)} dy = \frac{(1-\gamma-\sqrt{x})^2}{2},$$

where  $0 \leq y \leq (1-\gamma)^2$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{(1-\gamma)^2} \left( \frac{(1-\gamma-\sqrt{x})^2}{2} - \frac{(1-\gamma)^2}{5} \right)^2 \frac{6(1-\gamma-\sqrt{x})^2}{(1-\gamma)^4} dx \\ &= \frac{19(1-\gamma)^4}{1400}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{9(1-\gamma)^4}{350},$$

one obtains that the probabilistic correlation of  $Y_\gamma$  with respect to  $X_\gamma$  is,

$$\eta^2(Y_\gamma|X_\gamma) = \frac{19}{36}.$$

That is, the  $f$ -weighted possibilistic correlation ratio of  $B$  with respect to  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{19}{36} f(\gamma) d\gamma = \frac{19}{36}.$$

### 5.3 Discussion

In this chapter a correlation ratio is introduced for marginal possibility distributions of joint possibility distributions and this new principle is illustrated by five examples. This new concept has properties similar to the probabilistic correlation ratio. One possible direction for applications is sensitivity analysis. The main drawback of the correlation ratio is that there is an inherent problem with computing (or even estimating) it in a simple manner: the calculation of the conditional expectation is not a straightforward task even in the simplest situations. If this problem of estimation can be solved in an efficient way, the applications for example in portfolio optimization would be promising. One possible way would be the generalization of the method described in [62].

## Chapter 6

# Quasi Fuzzy Numbers

A *quasi fuzzy number*  $A$  is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions [10]

$$\lim_{t \rightarrow \infty} \mu_A(t) = 0, \quad \lim_{t \rightarrow -\infty} \mu_A(t) = 0.$$

A *quasi triangular* fuzzy number (see Fig. 6.1) is a quasi fuzzy number with a unique maximizing point. Furthermore, we call  $\mathcal{Q}$  the family of all quasi fuzzy numbers. Quasi fuzzy numbers can also be considered as possibility distributions [29]. If  $A$  is a quasi fuzzy number, then  $[A]^\gamma$  is a closed convex (compact) subset of  $\mathbb{R}$  for any  $\gamma > 0$ .  $a_1(\gamma)$  denotes the left-hand side and  $a_2(\gamma)$  denotes the right-hand side of the  $\gamma$ -cut, of  $A$  for any  $\gamma \in [0, 1]$ .

Different weighting functions can give different (case-dependent) importances to  $\gamma$ -levels sets of quasi fuzzy numbers. It is motivated in part by the desire to give less importance to the lower levels of fuzzy sets [41] (it is why the weighting function  $f$  should be monotone increasing).

### 6.1 Possibilistic Mean Value, Variance, Covariance and Correlation of Quasi Fuzzy Numbers

The possibilistic mean (or expected value), variance and covariance can be defined from the measure of possibilistic interactivity (as shown in [13, 39, 40]). If  $f(\gamma) = 1$  for all  $\gamma \in [0, 1]$  then we get from 3.1

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma.$$

In this chapter the *natural* weighting function,  $f(\gamma) = 2\gamma$ , will be used. In this case

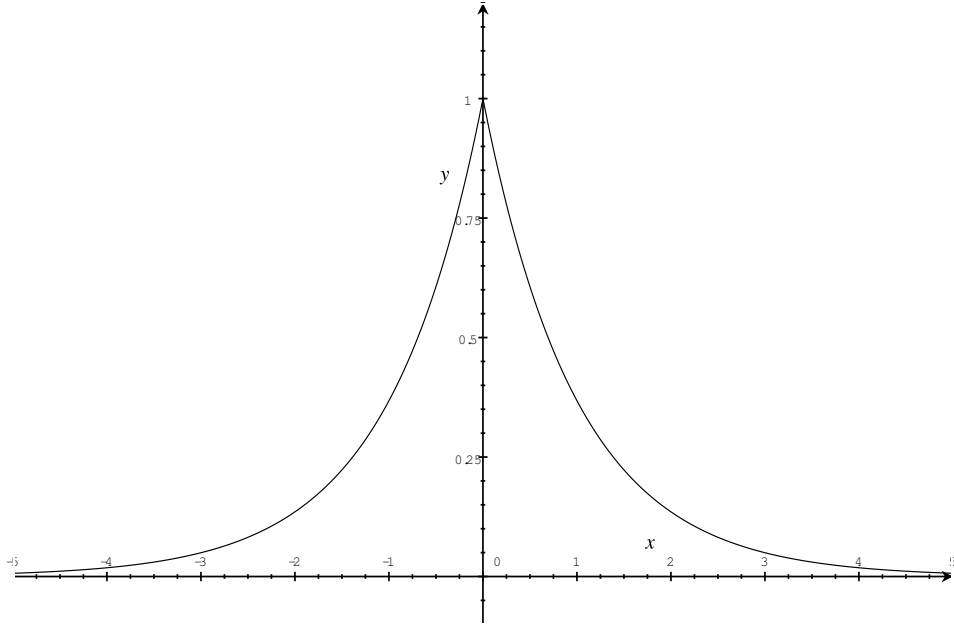


Figure 6.1: A quasi triangular fuzzy number with membership function  $e^{-|x|}$ .

the possibilistic mean value is defined by,

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma \\ &= \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) d\gamma, \end{aligned} \tag{6.1}$$

which is the definition originally introduced by Carlsson and Fullér in 2001 [9]. We note here that from the equality

$$\begin{aligned} E(A) &= \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) d\gamma \\ &= \frac{\int_0^1 \gamma \cdot \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma}{\int_0^1 \gamma d\gamma}, \end{aligned}$$

it follows that  $E(A)$  is nothing else but the level-weighted average of the arithmetic means of all  $\gamma$ -level sets, that is, the weight of the arithmetic mean of  $a_1(\gamma)$  and  $a_2(\gamma)$  is just  $\gamma$ .

The concept of possibilistic mean value can be extended to the family of quasi fuzzy numbers:

**Definition 6.1.** The  $f$ -weighted possibilistic mean value of  $A \in \mathcal{Q}$  is defined as

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma,$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$  for all  $\gamma > 0$ . The value of  $E_f(A)$  does not depend on the boundedness of the support of  $A$ .

The possibilistic mean value is originally defined for fuzzy numbers (i.e. quasi fuzzy numbers with bounded support). If the support of a quasi fuzzy number  $A$  is unbounded then its possibilistic mean value might not exist. However, for a symmetric quasi fuzzy number,  $A$ , it is easy to see that  $E_f(A) = a$ , where  $a$  is the center of symmetry, for any weighting function  $f$ .

In the following a family of quasi fuzzy numbers will be characterized for which it is possible to calculate the possibilistic mean value. First an example is shown for a quasi triangular fuzzy number that does not have a mean value.

**Example 6.1.** Consider the following quasi triangular fuzzy number

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{\sqrt{x+1}} & \text{if } 0 \leq x \end{cases}$$

In this case

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma^2} - 1,$$

and its possibilistic mean value can not be computed, since the following integral does not exist (not finite),

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 \left( \frac{1}{\gamma^2} - 1 \right) \gamma d\gamma = \int_0^1 \left( \frac{1}{\gamma} - \gamma \right) d\gamma.$$

This example is very important: if the membership function of the quasi fuzzy number tends to zero slower than the function  $1/\sqrt{x}$  then it is not possible to calculate the possibilistic mean value, (clearly, the value of the integral will be infinite), otherwise the possibilistic mean value does exist.

To show this, suppose that there exists  $\varepsilon > 0$ , such that the membership function of quasi fuzzy number  $A$  satisfies the property,

$$\mu_A(x) = O(x^{-\frac{1}{2}-\varepsilon})$$

if  $x \rightarrow +\infty$ . This means that there exist  $M$  and  $x_0 \in \mathbb{R}$  such that,

$$\mu_A(x) \leq Mx^{-\frac{1}{2}-\varepsilon},$$

if  $x > x_0$  and where  $M$  is a positive real number. Thus the possibilistic mean value of  $A$  is bounded from above by

$$M^{-\frac{1}{2}-\varepsilon}$$

multiplied by the possibilistic mean value of a quasi fuzzy number with membership function  $x^{-\frac{1}{2}-\varepsilon}$  plus an additional constant (because of the properties of a quasi fuzzy number, the interval  $[0, x_0]$  accounts for a finite value in the integral).

Suppose that,

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \\ x^{-\frac{1}{2}-\varepsilon} & \text{if } x \geq 1 \end{cases}$$

A similar reasoning holds for negative fuzzy numbers with membership function  $(-x)^{-\frac{1}{2}-\varepsilon}$ . Then we get,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \gamma^{-\frac{1}{2}-\varepsilon},$$

and since

$$\frac{\varepsilon - \frac{1}{2}}{\varepsilon + \frac{1}{2}} \neq 1,$$

we can calculate the possibilistic mean value of  $A$  as,

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 \gamma^{-\frac{1}{\varepsilon+\frac{1}{2}}} \gamma d\gamma \\ &= \int_0^1 \gamma^{-\frac{\varepsilon-\frac{1}{2}}{\varepsilon+\frac{1}{2}}} d\gamma = \left(\varepsilon + \frac{1}{2}\right) \left[ \gamma^{\frac{1}{\varepsilon+\frac{1}{2}}} \right]_0^1 = \varepsilon + 1/2 \end{aligned}$$

**Theorem 6.1.** *If  $A$  is a non-symmetric quasi fuzzy number then  $E_f(A)$  exists if and only if there exist real numbers  $\varepsilon, \delta > 0$ , such that,*

$$\mu_A(x) = O(x^{-\frac{1}{2}-\varepsilon}),$$

if  $x \rightarrow +\infty$  and

$$\mu_A(x) = O((-x)^{-\frac{1}{2}-\delta}),$$

if  $x \rightarrow -\infty$ .

**Note 6.1.** *If one considers other weighting functions, a sufficient and necessary condition is  $\mu_A(x) = O(x^{-1-\varepsilon})$ , when  $x \rightarrow +\infty$  (in the worst case, when  $f(\gamma) = 1$ ,  $\frac{1}{\gamma}$  is the critical growth rate.)*

**Example 6.2.** *Consider the following quasi triangular fuzzy number (depicted in Fig. 6.2),*

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{x+1} & \text{if } 0 \leq x \end{cases}$$



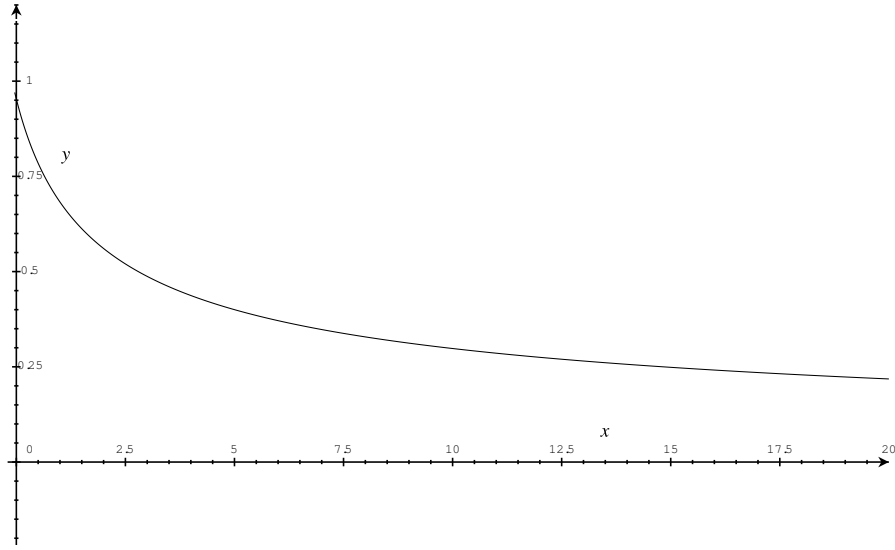


Figure 6.2: Quasi triangular fuzzy number  $1/(x+1), x \geq 0$ .

In this case

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma} - 1,$$

and its possibilistic mean value is,

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 \left( \frac{1}{\gamma} - 1 \right) \gamma d\gamma \\ &= \int_0^1 (1 - \gamma) d\gamma = 1/2. \end{aligned}$$

This example is very important: the volume of  $A$  can not be normalized since  $\int_0^\infty \mu_A(x) dx$  does not exist. In other words,  $\mu_A$  can not be considered as a density function of any random variable.

The measure of  $f$ -weighted possibilistic variance of a fuzzy number  $A$  is the  $f$ -weighted average of the probabilistic variances of the respective uniform distributions on the level sets of  $A$ . That is, the  $f$ -weighted *possibilistic variance* of  $A$  is defined by 3.3

$$\text{Var}_f(A) = \int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

The concept of possibilistic variance can be extended to the family of quasi fuzzy numbers:

**Definition 6.2.** The measure of  $f$ -weighted possibilistic variance of a quasi fuzzy number  $A$  is the  $f$ -weighted average of the probabilistic variances of the respective uniform

distributions on the level sets of  $A$ . That is, the  $f$ -weighted possibilistic variance of  $A$  is defined by

$$\text{Var}_f(A) = \int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$  for all  $\gamma > 0$ . The value of  $\text{Var}_f(A)$  does not depend on the boundedness of the support of  $A$ . If  $f(\gamma) = 2\gamma$  then the notation  $\text{Var}(A)$  will be used.

From the definition it follows that in this case there can be no distinction made between the symmetric and non-symmetric case. And it is also obvious, because of the square of the  $a_1(\gamma)$  and  $a_2(\gamma)$  functions in the definition, that the decreasing rate of the membership function has to be the square of the rate determined in case of the mean value. One can conclude:

**Theorem 6.2.** *If  $A$  is a quasi fuzzy number then  $\text{Var}(A)$  exists if and only if there exist real numbers  $\varepsilon, \delta > 0$ , such that*

$$\mu_A(x) = O(x^{-1-\varepsilon})$$

*if  $x \rightarrow +\infty$  and*

$$\mu_A(x) = O((-x)^{-1-\delta}),$$

*if  $x \rightarrow -\infty$ .*

**Note 6.2.** *When considering other weighting functions, one needs to require that*

$$\mu_A(x) = O(x^{-2-\varepsilon}),$$

*when  $x \rightarrow +\infty$  (in the worst case, when  $f(\gamma) = 1$ ,  $\frac{1}{\sqrt{\gamma}}$  is the critical growth rate.)*

**Example 6.3.** *Consider again the quasi triangular fuzzy number,*

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{x+1} & \text{if } 0 \leq x \end{cases}$$

*In this case we have,*

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma} - 1,$$

*and its possibilistic variance does not exist since*

$$\int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} 2\gamma d\gamma = \int_0^1 \frac{(1/\gamma - 1)^2}{12} 2\gamma d\gamma = \infty.$$

The  $f$ -weighted *measure of possibilistic covariance* between  $A, B \in \mathcal{F}$ , (with respect to their joint distribution  $C$ ), defined by 3.2, can be written as

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\text{cov}(X_\gamma, Y_\gamma)$  denotes their probabilistic covariance.

**Definition 6.3.** *The  $f$ -weighted measure of possibilistic covariance between  $A, B \in \mathcal{Q}$ , (with respect to their joint distribution  $C$ ), is defined by,*

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for any  $\gamma > 0$ .

It is easy to see that the possibilistic covariance is an absolute measure in the sense that it can take any value from the real line.

The concept of possibilistic correlation can be extended in a similar way to the family of quasi fuzzy numbers.

**Definition 6.4.** *The  $f$ -weighted possibilistic correlation coefficient of  $A, B \in \mathcal{Q}$  (with respect to their joint distribution  $C$ ) is defined by*

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}}$$

and, where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for any  $\gamma > 0$ .

## 6.2 Probability versus Possibility: The Case of Exponential Function

In this section the possibilistic mean value and variance of a quasi triangular fuzzy number defined by the membership function  $e^{-x}$ ,  $x \geq 0$ , will be investigated. It can also be seen as a density function of a standard exponential random variable (see Fig. 6.3). In probability theory and statistics, the exponential distribution is a family of continuous probability distributions. It describes the time between events in a *Poisson process*, i.e. a process in which events occur continuously and independently at a constant average rate.

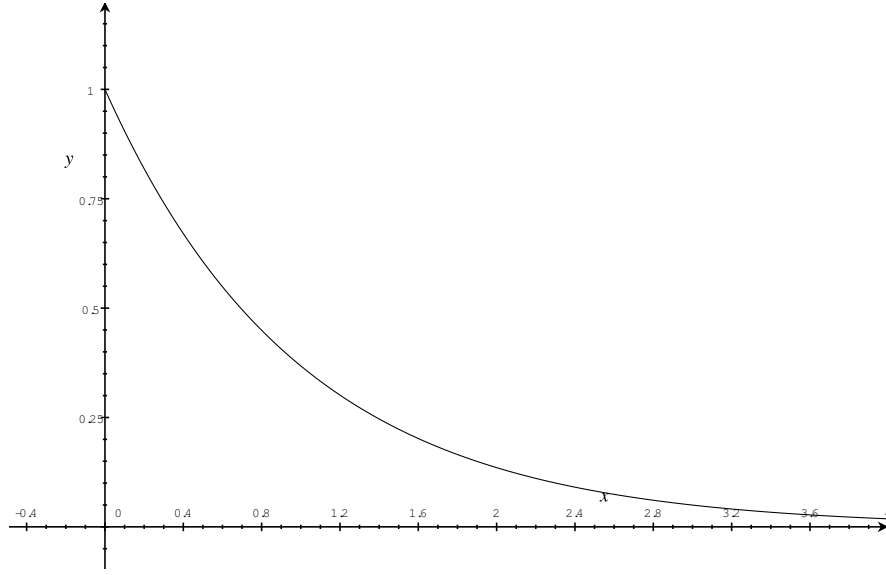


Figure 6.3: Quasi triangular fuzzy number and density function of an exponential random variable with parameter one:  $e^{-x}$ ,  $x \geq 0$ .

Consider the following quasi triangular fuzzy number

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0. \end{cases}$$

From  $\int_0^\infty \mu_A(x)dx = 1$  it follows that  $\mu_A$  can also be considered as the density function of a standard exponential random variable (with parameter one). It is well-known that the mean value and the variance of this probability distribution is equal to one. In the fuzzy case,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = -\ln \gamma,$$

its possibilistic mean value is

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 -(\ln \gamma)\gamma d\gamma = \frac{1}{4},$$

and its possibilistic variance is,

$$\begin{aligned} \text{Var}(A) &= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} 2\gamma d\gamma \\ &= \int_0^1 \frac{(-\ln \gamma)^2}{6} \gamma d\gamma = \frac{1}{24}. \end{aligned}$$

Let  $C$  be the joint possibility distribution, defined by the membership function,

$$\mu_C(x, y) = e^{-(x+y)}, \quad x \geq 0, y \geq 0,$$

of quasi fuzzy numbers  $A$  and  $B$  with membership functions

$$\mu_A(x) = e^{-x}, x \geq 0, \quad \text{and} \quad \mu_B(y) = e^{-y}, y \geq 0.$$

In other words, the membership function of  $C$  is defined by a simple multiplication (by Larsen t-norm [59]) of the membership values of  $\mu_A(x)$  and  $\mu_B(y)$ , that is,  $\mu_C(x, y) = \mu_A(x) \times \mu_B(y)$ . The  $\gamma$ -cut of  $C$  can be computed by

$$[C]^\gamma = \{(x, y) \mid x + y \leq -\ln \gamma; x, y \geq 0\}.$$

Then

$$M(X_\gamma) = M(Y_\gamma) = -\frac{\ln \gamma}{3},$$

$$M(X_\gamma^2) = M(Y_\gamma^2) = \frac{(\ln \gamma)^2}{6},$$

and,

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(\ln \gamma)^2}{6} - \frac{(\ln \gamma)^2}{9} = \frac{(\ln \gamma)^2}{18}.$$

Similarly,

$$\text{var}(Y_\gamma) = \frac{(\ln \gamma)^2}{18}.$$

Furthermore,

$$M(X_\gamma Y_\gamma) = \frac{(\ln \gamma)^2}{12},$$

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(\ln \gamma)^2}{36}.$$

The probabilistic correlation can be computed as

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

That is,  $\rho(X_\gamma, Y_\gamma) = -1/2$  for any  $\gamma > 0$ . Consequently, their possibilistic correlation coefficient is,  $\rho_f(A, B) = -1/2$  for any weighting function  $f$ .

On the other hand, in a probabilistic context,  $\mu_C(x, y) = \mu_A(x) \times \mu_B(y) = e^{-(x+y)}$  can be also considered as the joint density function of independent exponential marginal probability distributions with parameter one. That is, in a probabilistic context, their (probabilistic) correlation coefficient is equal to zero.

**Note 6.3.** *The probabilistic correlation coefficient between two standard exponential marginal probability distributions cannot go below  $(1 - \pi^2/6)$ . Really, the lower limit, denoted by  $\tau$ , can be computed from,*

$$\tau = \int_0^\infty \int_0^\infty (1 - e^{-x} - e^{-y})^+ - (1 - e^{-x})(1 - e^{-y}) dx dy$$

$$\begin{aligned}
&= - \int_0^\infty \int_0^\infty e^{-x} e^{-y} dx dy + \iint_{0 < x, 0 < y, 1 < e^{-x} + e^{-y}} (1 - e^{-x} - e^{-y})^+ dx dy \\
&= -1 + \iint_{0 < x, 0 < y, 1 < e^{-x} + e^{-y}} (2e^{-x} - 1) dx dy
\end{aligned}$$

using the substitutions  $u = e^{-x}, v = e^{-y}$ ,

$$\begin{aligned}
\tau &= -1 + \iint_{u < 1, v < 1, u+v > 1} \left( \frac{2}{u} - \frac{1}{uv} \right) du dv = -1 + \int_0^1 \frac{1}{u} \int_0^1 \left( 2 - \frac{1}{v} \right) dv du \\
&= 1 + \int_0^1 \frac{2u + \log(1-u)}{u} du = -1 + \int_0^1 \frac{\log(1-u)}{u} du \\
&= \int_0^1 \sum_{k=1}^{\infty} \frac{u^{k-1}}{k} du = 1 - \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 - \frac{\pi^2}{6}.
\end{aligned}$$

In the case of possibility distributions there is no known lower limit [44].

If the joint possibility distribution  $C$  is given by the minimum operator (Mamdani t-norm [69]),

$$\mu_C(x, y) = \min\{\mu_A(x), \mu_B(y)\} = \min\{e^{-x}, e^{-y}\},$$

$x \geq 0, y \geq 0$ , then  $A$  and  $B$  are non-interactive marginal possibility distributions and, therefore, their possibilistic correlation coefficient equals to zero.

### 6.3 Discussion

The principles of possibilistic mean value, variance, covariance and correlation of fuzzy numbers were generalized to a more general class of fuzzy quantities: to quasi fuzzy numbers. Some conditions were shown for the existence of possibilistic mean value and variance for quasi fuzzy numbers.

## Chapter 7

# The Fuzzy Pay-off Method

Real option valuation (ROV) is based on the observation that the possibilities financial options give their holder resemble the possibilities to invest in real investments and possibilities found within real investments, i.e., managerial flexibility - “an irreversible investment opportunity is much like a financial call option” [80]. In other words, real option valuation handles investment opportunities as options and values them with option valuation models. Real options are useful as a model for strategic and operational decision-making, and as a valuation and numerical analysis tool. This chapter focuses on the use of real options in quantitative analysis, and particularly on the derivation of the real option value for a given investment opportunity.

Real options are commonly valued with the same methods that have been used to value financial options and most of the methods are complex and demand a good understanding of the underlying mathematics, issues that make their use difficult in practice. Since all of the traditional methods were designed to value financial options, when using them in real option context, the decision makers have to be aware of the significant differences between financial and real options (as it was shortly described in Chapter 3).

A novel approach to real option valuation was presented in [21], [71], and in [22], where the real option value is calculated from a pay-off distribution, derived from a probability distribution of the NPV for a project that is generated with a (Monte-Carlo) simulation. The authors show that the results from the method converge to the results from the analytical Black-Scholes method. The method presented greatly simplifies the calculation of the real option value. The method does not suffer from the problems associated with the assumptions connected to the market processes and used in the Black-Scholes and the binomial option valuation methods. The method utilizes cash-flow scenario based estimation of the future outcomes to derive the future pay-off distribution the way profitability analysis is commonly done in companies.

Uncertainty in the financial investment context means that it is in practice impossible to give absolutely correct estimates of, e.g., future cash-flows. There may be a number of reasons for this, see, e.g., [50].

Fuzzy sets can be used to formalize inaccuracy that exists in human decision making and as a representation of vague, uncertain or imprecise knowledge, e.g., “a future cash flow at year ten is about x euro”. “Fuzzy set-based methodologies blur the traditional line between qualitative and quantitative analysis, since the modeling may reflect more the type of information that is available rather than researchers’ preferences” [92] and indeed in economics “the use of fuzzy subsets theory leads to results that could not be obtained by classical methods” [81].

To estimate future cash flows and discount rates “One usually employs educated guesses, based on expected values or other statistical techniques” [8], which is consistent with the use of fuzzy numbers. In practical applications the most used fuzzy numbers are trapezoidal and triangular fuzzy numbers. They are used, because they make many operations possible and are intuitively understandable and interpretable.

Fuzzy numbers (fuzzy logic) have been adopted to option valuation models in (binomial) pricing an option with a fuzzy payoff, e.g., in [74], and in Black-Scholes valuation of financial options in, e.g., [104]. There are also some option valuation models that present a combination of probability theory and fuzzy sets, e.g., [115]. Fuzzy numbers have also been applied to the valuation of real options in, e.g., [11], [16], and [12]. More recently there are a number of papers that present the application of fuzzy RO models in the industry setting, e.g., [15]; [93]. There are also specific fuzzy models for the analysis of the value of optionality for very large industrial real investments, e.g., [17].

In the following a new method for valuation of real options using fuzzy numbers is presented. The model is based on the previous literature on real option valuation, especially the findings in [21]. The use of the method is illustrated with a selection of different types of fuzzy numbers. Then the concept of credibility measure is recalled and incorporated into the model. Finally, a real world example on patent valuation is provided.

## **7.1 New Fuzzy Pay-Off Method for Valuation of Real Options from Fuzzy Numbers**

[71] and [21] present a practical probability theory based method for the calculation of real option value (ROV) and show that the method and results from the method are mathematically equivalent to the Black-Scholes formula [3]. The method is based on simulation generated probability distributions for the NPV of future project outcomes. The method implies that: “the real-option value can be understood simply as the average net profit appropriately discounted to Year 0, the date of the initial R & D investment decision, contingent on terminating the project if a loss is forecasted at the future launch decision date”. The project outcome probability distributions are used to generate a payoff distribution, where the negative outcomes (subject to terminating the project) are truncated into one chunk that will cause a zero payoff, and where the probability weighted average value of the resulting payoff distribution is the real option value.

Fuzzy numbers can be used in representing the expected future distribution of possi-



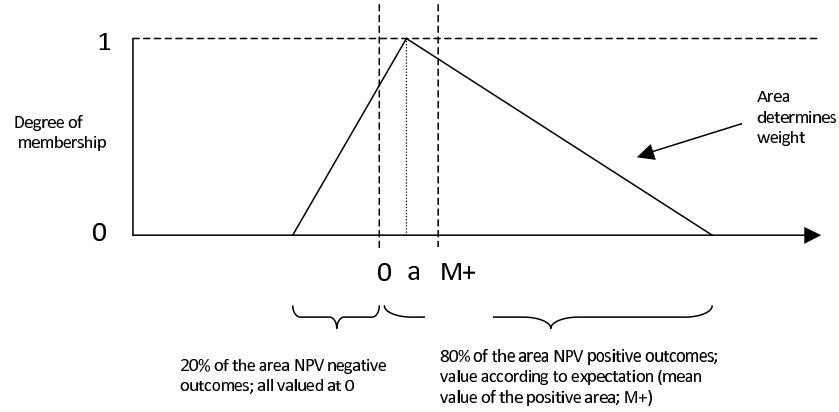


Figure 7.1: Triangular fuzzy number (a possibility distribution), defined by three points  $[a, \alpha, \beta]$  describing the NPV of a prospective project; (20% and 80% are for illustration purposes only)

ble project costs and revenues, and hence also the profitability (NPV) outcomes. When using fuzzy numbers the fuzzy NPV itself is the payoff distribution from the project.

The method presented in [21] implies that the weighted average of the positive outcomes of the payoff distribution is the real option value; in the case with fuzzy numbers, the weighted average is the possibilistic mean value of the positive NPV outcomes.

Real option value is calculated from the fuzzy NPV as:

$$FROV = \frac{A(\text{positive})}{A(\text{total})} \times \text{Fuzzy mean}(\text{positive NPV side}) \quad (7.1)$$

It is easy to see that when the whole fuzzy number is above zero, the ROV is the possibilistic mean of the number, and when the whole fuzzy number is below zero, the ROV is 0.

The new method is based on the observations that real option value is the probability weighted average of the positive values of a payoff distribution of a project, which is nothing more than the fuzzy NPV of the project, and that for fuzzy numbers the probability weighted average of the positive values of the payoff distribution is nothing else than the weighted possibilistic mean of the positive values of the fuzzy NPV.

## 7.2 Calculating the ROV with the Fuzzy Pay-Off Method with a Selection of Different Types of Fuzzy Numbers

As the form of a fuzzy number may vary the most used forms are the triangular and trapezoidal fuzzy numbers. First, the positive area and the possibilistic mean of the positive area of a triangular fuzzy pay-off  $A = (a, \alpha, \beta)$  are calculated in the case of  $a - \alpha < 0 < a$ . Variable  $z$ , where  $0 \leq z \leq \alpha$ , represents the distance of a general cut

point from  $a - \alpha$  at which the triangular fuzzy number (distribution) is divided into two parts - for this purpose the variable  $z$  gets the value  $\alpha - a$  (to calculate the positive part of  $A$ ). The notation

$$(A|z)(t) = \begin{cases} 0 & \text{if } t \leq a - \alpha + z \\ A(t) & \text{otherwise} \end{cases}$$

is used for the membership function of the right-hand side of a triangular fuzzy number truncated at point  $a - \alpha + z$ , where  $0 \leq z \leq \alpha$ .

Then the possibilistic mean value of this truncated triangular fuzzy number can be calculated as

$$E(A|z) = I_1 + I_2 = \int_0^{z_1} \gamma(2a - \alpha + z + (1 - \gamma)\beta) d\gamma + \int_{z_1}^1 \gamma(2a - (1 - \gamma)\alpha + (1 - \gamma)\beta) d\gamma \quad (7.2)$$

where

$$z_1 = 1 - \frac{\alpha - z}{\alpha} = \frac{z}{\alpha}$$

After computing the integrals, one obtains the following:

$$z_1 = 1 - \frac{\alpha - z}{\alpha} = \frac{z}{\alpha}$$

$$I_1 = \int_0^{z_1} [(2a - \alpha + z + \beta)\gamma - \beta\gamma^2] d\gamma = (2a - \alpha + z + \beta) \frac{z^2}{2\alpha^2} - \beta \frac{z^3}{3\alpha^3}$$

$$I_2 = \int_{z_1}^1 [(2a + \beta - \alpha)\gamma - \gamma^2(\beta - \alpha)] d\gamma = (2a + \beta - \alpha) \left( \frac{1}{2} - \frac{z^2}{2\alpha^2} \right) - (\beta - \alpha) \left( \frac{1}{3} - \frac{z^3}{3\alpha^3} \right)$$

$$I_1 + I_2 = (2a - \alpha + z + \beta) \times \frac{z^2}{2\alpha^2} - \beta \times \frac{z^3}{3\alpha^3} + (2a + \beta - \alpha) \times \left( \frac{1}{2} - \frac{z^2}{2\alpha^2} \right) -$$

$$(\beta - \alpha) \times \left( \frac{1}{3} - \frac{z^3}{3\alpha^3} \right) = \frac{z^3}{2\alpha^2} + \frac{2a - \alpha + \beta}{2} + \frac{\alpha - \beta}{3} - \alpha \frac{z^3}{3\alpha^3}$$

$$E = \frac{z^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}$$

To derive the real option value with the above formulas, the ratio between the positive area of the triangular fuzzy number and the total area of the same number has to be calculated and multiplied with the possibilistic mean value of the positive part of the number ( $E$ ), according to what is depicted in Fig. 7.1.

In the following, the possibilistic mean value is calculated for the positive part of a trapezoidal fuzzy pay-off distribution and the same for a special form of fuzzy pay-off distribution.

A trapezoidal fuzzy pay-off distribution is defined as

$$A(u) = \begin{cases} \frac{u}{\alpha} - \frac{a_1 - \alpha}{\alpha} & \text{if } a_1 - \alpha \leq u \leq a_1 \\ 1 & \text{if } a_1 \leq u \leq a_2 \\ \frac{u}{-\beta} + \frac{a_2 + \beta}{\beta} & \text{if } a_2 \leq u \leq a_2 + \beta \\ 0 & \text{otherwise} \end{cases}$$

with  $\gamma$ -level sets  $[A]^\gamma = [\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta]$ . The possibilistic mean value of  $A$  is the following:

$$\begin{aligned} E(A) &= \int_0^1 \gamma(\gamma\alpha + a_1 - \alpha - \gamma\beta + a_2 + \beta) d\gamma \\ &= (a_1 - \alpha + a_2 + \beta) \frac{1}{2} + (\beta - \alpha) \frac{1}{3} \\ &= \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6} \end{aligned}$$

The mean value has to be calculated in 5 different cases depending on the position of  $z$ :

$$1. \ z < a_1 - \alpha : E(A \mid z) = E(A).$$

$$2. \ a_1 - \alpha < z < a_1 : \gamma_z = \frac{z}{\alpha} - \frac{a_1 - \alpha}{\alpha},$$

$$[A]^\gamma = \begin{cases} (z, -\gamma\beta + a_2 + \beta) & \text{if } \gamma \leq \gamma_z \\ (\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta) & \text{if } \gamma_z \leq \gamma \leq 1, \end{cases}$$

$$\begin{aligned} E(A \mid z) &= \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta) d\gamma + \int_{\gamma_z}^1 \gamma(\gamma\alpha + a_1 - \alpha - \gamma\beta + a_2 + \beta) d\gamma \\ &= \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6} + (z - a_1 + \alpha) \frac{\gamma_z^2}{2} - \alpha \frac{\gamma_z^3}{3}. \end{aligned}$$

$$3. \ a_1 < z < a_2 : \gamma_z = 1,$$

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta],$$

$$E(A \mid z) = \int_0^1 \gamma(z - \gamma\beta + a_2 + \beta) d\gamma = \frac{z + a_2}{2} + \frac{\beta}{6}.$$

$$4. \ a_2 < z < a_2 + \beta : \gamma_z = \frac{z}{-\beta} + \frac{a_2 + \beta}{\beta},$$

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta], \text{ if } \gamma < \gamma_z,$$

$$E(A \mid z) = \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta) d\gamma = (z + a_2 + \beta) \frac{\gamma_z^2}{2} - \beta \frac{\gamma_z^3}{3}$$

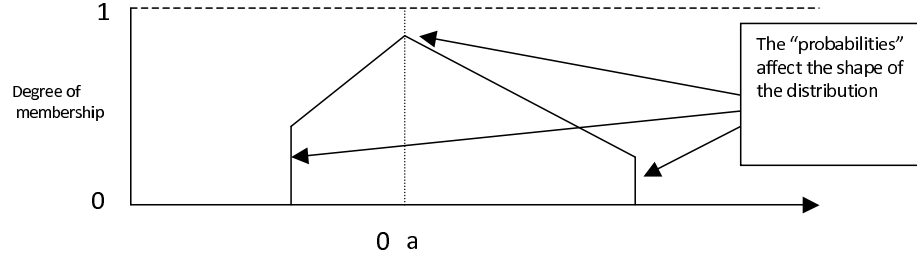


Figure 7.2: Illustration of the special case

5.  $a_2 + \beta < z : E(A | z) = 0$ .

In the case depicted in Fig. 7.2, the managers have already performed the analysis of three scenarios and have assigned probabilities to each scenario (adding to 100%). These 'probabilities' are assigned to the scenarios to obtain a fuzzy set:

$$A(u) = \begin{cases} (\gamma_3 - \gamma_1)\frac{u}{\alpha} - (\gamma_3 - \gamma_1)\frac{a - \alpha}{\alpha} + \gamma_1 & \text{if } a - \alpha \leq u \leq a \\ \gamma_3 & \text{if } u = a \\ (\gamma_2 - \gamma_3)\frac{u}{\beta} - (\gamma_2 - \gamma_3)\frac{a}{\beta} + \gamma_3 & \text{if } a \leq u \leq a + \beta \\ 0 & \text{otherwise} \end{cases}$$

The fuzzy mean value can be calculated as:

$$E(A) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma = \int_0^1 \gamma a_1(\gamma)d\gamma + \int_0^1 \gamma a_2(\gamma)d\gamma$$

where

$$\begin{aligned} \int_0^1 \gamma a_1(\gamma)d\gamma &= \int_0^{\gamma_1} \gamma(a - \alpha)d\gamma + \int_{\gamma_1}^{\gamma_3} \gamma\left(\frac{\gamma - \gamma_1}{\gamma_3 - \gamma_1}\alpha + a - \alpha\right)d\gamma \\ &= (a - \alpha)\frac{\gamma_1^2}{2} + \left(a - \alpha - \frac{\alpha\gamma_1}{\gamma_3 - \gamma_1}\right)\left(\frac{\gamma_3^2}{2} - \frac{\gamma_1^2}{2}\right) + \frac{\alpha}{\gamma_3 - \gamma_1}\left(\frac{\gamma_3^3}{3} - \frac{\gamma_1^3}{3}\right) \\ \int_0^1 \gamma a_2(\gamma)d\gamma &= \int_0^{\gamma_2} \gamma(a + \beta)d\gamma + \int_{\gamma_2}^{\gamma_3} \gamma\left(\frac{\gamma - \gamma_3}{\gamma_2 - \gamma_3}\beta + a\right)d\gamma \\ &= (a + \beta)\frac{\gamma_2^2}{2} + \left(a - \frac{\beta\gamma_3}{\gamma_2 - \gamma_3}\right)\left(\frac{\gamma_3^2}{2} - \frac{\gamma_2^2}{2}\right) + \frac{\beta}{\gamma_2 - \gamma_3}\left(\frac{\gamma_3^3}{3} - \frac{\gamma_2^3}{3}\right) \end{aligned}$$

$$\begin{aligned} E(A) &= \frac{\gamma_1^2}{2} \frac{\alpha\gamma_1}{\gamma_3 - \gamma_1} + \frac{\gamma_2^2}{2} \left(\beta + \frac{\beta\gamma_3}{\gamma_2 - \gamma_3}\right) + \frac{\gamma_3^2}{2} \left(2a - \alpha - \frac{\alpha\gamma_1}{\gamma_3 - \gamma_1} - \frac{\beta\gamma_3}{\gamma_2 - \gamma_3}\right) - \frac{\gamma_1^3}{3} \frac{\alpha}{\gamma_3 - \gamma_1} \\ &\quad - \frac{\gamma_2^3}{3} \frac{\beta}{\gamma_2 - \gamma_3} + \frac{\gamma_3^3}{3} \left(\frac{\alpha}{\gamma_3 - \gamma_1} + \frac{\beta}{\gamma_2 - \gamma_3}\right) \end{aligned}$$

The mean value has to be calculated in 4 different cases depending on the position of  $z$ :

$$1. \ z < a - \alpha : E(A \mid z) = E(A).$$

$$2. \ a - \alpha < z < a : \gamma_z = (\gamma_3 - \gamma_1) \frac{z}{\alpha} - (\gamma_3 - \gamma_1) \frac{a - \alpha}{\alpha} + \gamma_1,$$

$$\begin{aligned} E(A \mid z) &= \frac{\gamma_z^2}{2} \left( z - a + \alpha + \frac{\alpha \gamma_1}{\gamma_3 - \gamma_1} \right) + \frac{\gamma_z^2}{2} \left( \beta + \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) + \\ &\quad \frac{\gamma_z^2}{2} \left( 2a - \alpha - \frac{\alpha \gamma_1}{\gamma_3 - \gamma_1} - \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) - \frac{\gamma_z^3}{3} \frac{\alpha}{\gamma_3 - \gamma_1} \\ &\quad - \frac{\gamma_z^3}{3} \frac{\beta}{\gamma_2 - \gamma_3} + \frac{\gamma_z^3}{3} \left( \frac{\alpha}{\gamma_3 - \gamma_1} + \frac{\beta}{\gamma_2 - \gamma_3} \right). \end{aligned}$$

$$3. \ a < z < a + \beta : \gamma_z = (\gamma_2 - \gamma_3) \frac{z}{\beta} - (\gamma_2 - \gamma_3) \frac{a}{\beta} + \gamma_3,$$

$$E(A \mid z) = \frac{\gamma_z^2}{2} \left( z + a - \frac{\beta}{\gamma_2 - \gamma_3} \right) + \frac{\gamma_z^2}{2} \left( \beta + \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) + \frac{\gamma_z^3}{3} \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} - \frac{\gamma_z^3}{3} \frac{\beta}{\gamma_2 - \gamma_3}.$$

$$4. \ a + \beta < z : E(A \mid z) = 0.$$

### 7.3 Credibility measure

The self-duality of a probability distribution is an essential property: if one can estimate the probability of an event, the probability of the complement can be calculated straightforwardly. To give an example in the context of real options: if the probability that the value of an investment will increase in the following two months is  $p$ , then the probability that the value will decrease is  $1 - p$ . In practical problems, this provides a clear interpretation and facilitates the work of the decision makers. In contrast, if a triangular fuzzy number with center  $a$  is considered as a representation for the value of an investment, the possibility that this value is greater than  $a$  is 1, and the possibility that it is smaller than  $a$  is 1 as well. From a managerial perspective, this type of information is quite confusing and fails to provide useful support in the decision making process. To overcome this feature in real life applications, one possible way is to use a credibility measure instead of possibility. It preserves all the essential and useful properties of possibility (and necessity) and provides an easy interpretation of the results.

Although the concept of possibility measure [111] has been widely used, it has no self-duality property. This was the main motivation behind the concept of credibility measure which was first defined in [66], where the authors used this subclass of fuzzy measures to define the expected value of a fuzzy random variable  $\xi$ . Later, credibility theory was founded by Liu ([67]), and in ([63]) Li and Liu gave the following four axioms as a sufficient and necessary condition for a credibility measure ( $\Theta$  is a nonempty set and  $\mathcal{P}(\Theta)$  is the power set of  $\Theta$ ):

1.  $\text{Cr}\{\Theta\} = 1$
2.  $\text{Cr}$  is increasing, i.e.,  $\text{Cr}\{C\} \leq \text{Cr}\{D\}$  whenever  $C \subset D$
3.  $\text{Cr}$  is self-dual, i.e.,  $\text{Cr}\{C\} + \text{Cr}\{C^c\} = 1$  for any  $C \in \mathcal{P}(\Theta)$
4.  $\text{Cr}\{\bigcup_i C_i\} \wedge 0.5 = \sup_i \text{Cr}\{C_i\}$  for any  $\{C_i\}$  with  $\text{Cr}\{C_i\} \leq 0.5$ .

**Definition 7.1** (Credibility measure). *A set function,  $\text{Cr}$ , is called a credibility measure if it satisfies the first four axioms.*

It is easy to see that  $\text{Cr}\{\emptyset\} = 0$ , and that the credibility measure takes value between 0 and 1. It can also be proved that a credibility measure is subadditive ([67]):

$$\text{Cr}\{C \cup D\} \leq \text{Cr}\{C\} + \text{Cr}\{D\} \text{ for any } C, D \in \mathcal{P}(\Theta).$$

To establish the connection between a fuzzy variable and a credibility measure, both defined on the credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$ , a fuzzy variable,  $A$ , can be seen as a function from this space to the set of real numbers, and its membership function can be derived from the credibility measure by

$$\mu(x) = (2\text{Cr}\{A = x\}) \wedge 1, \quad x \in \mathbb{R}.$$

$\{A \in B\}$  is termed as a fuzzy event, where  $B$  is a set of real numbers.

However, in practice a fuzzy variable is specified by its membership function. In this case we can calculate the credibility of fuzzy events by the credibility inversion theorem([67]):

**Theorem 7.1.** *Let  $A$  be a fuzzy variable with membership function  $\mu$ . Then for any set  $B$  of real numbers, we have*

$$\text{Cr}\{A \in B\} = \frac{1}{2}(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)).$$

With this formula it is possible to interpret the credibility in terms of the possibility and necessity measure, since  $\sup_{x \in B} \mu(x)$  and  $1 - \sup_{x \in B^c} \mu(x)$  are nothing else but  $\text{Pos}(B)$  and  $\text{Nec}(B)$ , respectively. Using this two measures, the theorem can be formulated as

$$\text{Cr}\{B\} = \frac{1}{2}(\text{Pos}(B) + \text{Nec}(B)). \quad (7.3)$$

**Note 7.1.** *If the credibility measure is defined using the equation (7.3), then Li and Liu in [63] proved that this is equivalent to the definition using the four axioms.*

**Example 7.1.** *In this example, the credibility of events is calculated in case of a triangular fuzzy number with peak (or center)  $a$ , left width  $\alpha > 0$  and right width  $\beta > 0$ . From*

the definition of credibility measure, the credibility of the event  $\{A \leq x\}$  is the following:

$$\text{Cr}\{A \leq x\} = \begin{cases} 0 & \text{if } x \leq a - \alpha \\ \frac{1}{2} - \frac{a - x}{2\alpha} & \text{if } a - \alpha \leq x \leq a \\ \frac{1}{2} + \frac{x - a}{2\beta} & \text{if } a \leq x \leq a + \beta \\ 0 & \text{if } a + \beta \leq x \end{cases}$$

### 7.3.1 Expected value using the credibility measure

In [66], as the first application of the credibility measure, the authors proposed a novel concept of expected value for normalized fuzzy variables motivated by the theory of Choquet integrals (a fuzzy variable is said to be normalized, if there exists  $x_0 \in \mathbb{R}$  such that  $\mu_A(x_0) = 1$ ):

**Definition 7.2** (Expected value). *The expected value of a normalized fuzzy variable,  $\xi$ , is defined by*

$$E_c[\xi] = \int_0^\infty \text{Cr}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\} dr,$$

provided that at least one of the integrals is finite.

It is important to note that fuzzy numbers are normalized fuzzy variables by definition.

**Example 7.2.** *Let  $A = (a, \alpha, \beta)$  be a triangular fuzzy number. From the definition, the credibilistic expected value of  $A$  is obtained as:*

$$E_c[A] = a + \frac{\beta - \alpha}{4}.$$

*If  $A = (a_1, a_2, \alpha, \beta)$  is a trapezoidal fuzzy number defined by the membership function*

$$\mu_A(x) = \begin{cases} 1 - \frac{a_1 - x}{\alpha} & \text{if } a_1 - \alpha \leq x \leq a_1 \\ 1 & \text{if } a_1 \leq x \leq a_2 \\ 1 - \frac{x - a_2}{\beta} & \text{if } a_2 \leq x \leq a_2 + \beta \\ 0 & \text{otherwise,} \end{cases}$$

*then the credibilistic expected value of  $A$  is*

$$E_c[A] = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{4}.$$

Credibility theory and specifically the credibilistic expected value has been applied to problems from different areas: portfolio optimization ([113]), facility location problem in B2C e-commerce ([60]), transportation problems ([103]), logistics network design ([83]).

## 7.4 The Fuzzy Pay-off method with the credibilistic expectation

To use the credibility measure and the credibilistic expected value in this real option environment seems to be a natural choice. To compare the results with the possibilistic mean value, the same examples will be used. In case of credibilistic expected value, the calculation of the mean of the positive part is simply

$$E_c[A_+] = \int_0^\infty \text{Cr} \{A \geq r\} dr.$$

When the positive area and the mean of the positive area of a triangular fuzzy pay-off are calculated, we have to consider 4 cases:

- Case 1:  $0 < a - \alpha$ . In this case we have

$$E_c(A_+) = E_c(A) = a + \frac{\beta - \alpha}{4}.$$

**Note 7.2.** The possibilistic mean value of a triangular fuzzy number is  $E_p(A) = a + \frac{\beta - \alpha}{6}$ . Comparing this value to the result above, one can observe that

$$|E_p(A) - a| \leq |E_c(A) - a|.$$

Also important to note, that  $E_p(A) \leq E_c(A)$  if and only if the left width,  $\alpha$ , is smaller than the right width,  $\beta$ .

- Case 2:  $a - \alpha < 0 < a$ . Then the credibilistic expected value has the following form:

$$\begin{aligned} E_c[A_+] &= \int_0^\infty \text{Cr} \{A \geq r\} dr = \int_0^a \left(\frac{1}{2} + \frac{a-r}{2\alpha}\right) dr + \int_a^{a+\beta} \left(\frac{1}{2} - \frac{r-a}{2\beta}\right) dr \\ &= \frac{a}{2} + \frac{a^2}{4\alpha} + \frac{\beta}{4} \end{aligned}$$

- Case 3:  $a < 0 < a + \beta$ . In this case

$$E_c[A_+] = \int_0^\infty \text{Cr} \{A \geq r\} dr = \int_0^{a+\beta} \left(\frac{1}{2} - \frac{r-a}{2\beta}\right) dr = \frac{a}{2} + \frac{a^2}{4\beta} + \frac{\beta}{4}.$$

- Case 4:  $a + \beta < 0$ . Then it is easy to see that  $E(A_+) = 0$

If the NPV (pay-off) distribution is represented by a trapezoidal fuzzy number (with  $\gamma$ -level sets  $[A]^\gamma = [\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta]$ ), the credibility has the following form:



$$\text{Cr}\{A \leq r\} = \begin{cases} 0 & \text{if } r \leq a_1 - \alpha \\ \frac{1}{2} - \frac{a - x}{2\alpha} & \text{if } a_1 - \alpha \leq x \leq a_1 \\ \frac{1}{2} & \text{if } a_1 \leq x \leq a_2 \\ \frac{1}{2} + \frac{x - a}{2\beta} & \text{if } a_2 \leq x \leq a_2 + \beta \\ 0 & \text{if } a_2 + \beta \leq x \end{cases}$$

Then to calculate the credibilistic expected value for the positive part, we need to consider the following five cases:

- Case 1:  $0 < a_1 - \alpha$ . In this case we have  $E_c(A_+) = E_c(A)$ .

$$E_c(A_+) = E_c(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{4}.$$

**Note 7.3.** The possibilistic mean value of a trapezoidal fuzzy number is  $E_p(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6}$ . Comparing this value to the result above, it can be observed that

$$|E_p(A) - a| \leq |E_c(A) - a|.$$

Also important to note, that  $E_p(A) \leq E_c(A)$  if and only if the left width,  $\alpha$ , is smaller than the right width,  $\beta$ .

- Case 2:  $a_1 - \alpha < 0 < a_1$ . Then the credibilistic expected value can be calculated as:

$$\begin{aligned} E_c[A_+] &= \int_0^{a_1} \left(\frac{1}{2} + \frac{a_1 - r}{2\alpha}\right) dr + \int_{a_1}^{a_2} \frac{1}{2} dr + \int_{a_2}^{a_2 + \beta} \left(\frac{1}{2} - \frac{r - a_2}{2\beta}\right) dr \\ &= \frac{a_2}{2} + \frac{a_1^2}{4\alpha} + \frac{\beta}{4}. \end{aligned}$$

- Case 3:  $a_1 < 0 < a_2$ . In this case

$$E_c[A_+] = \int_0^{a_2} \frac{1}{2} dr + \int_{a_2}^{a_2 + \beta} \left(\frac{1}{2} - \frac{r - a_2}{2\beta}\right) dr = \frac{a_2}{2} + \frac{\beta}{4}.$$

- Case 4:  $a_2 < 0 < a_2 + \beta$ . In this case we have

$$E_c[A_+] = \int_0^{a_2 + \beta} \left(\frac{1}{2} - \frac{r - a_2}{2\beta}\right) dr = \frac{a_2}{2} + \frac{a_2^2}{4\beta} + \frac{\beta}{4}.$$

- Case 5:  $a_2 + \beta < z$ . Then it is easy to see that  $E(A|z) = 0$

## 7.5 Patent Valuation with the Fuzzy Pay-off Method

Patent valuation and analysis is gaining more and more interest as companies want to optimize their patent portfolios and are seeking for more efficient ways to do so [99, 19]. Much of IPR and patent analysis is conducted on qualitative basis, but there is an interest for methods that can provide qualitative analysis for patent valuation, see e.g., [84].

A general patent application process requires the following major outlays [7]:

- Initial local application fee.
- PCT filing fee at the  $x_1$  month, where  $x_1 < 12$  months.
- Major costs associated with internationalizing a patent application (in national phase) at the  $x_2$  month, where  $x_2 < 30$  months.

The follow-on commercial project has the following cash inflows and capital outlays: (i) expected cash inflows over the  $n$ -year period, which can be represented by  $CI_0, CI_1, \dots, CI_n$ ; (ii) expected outlays over the  $n$ -year period (including the outlay of year  $t$ ), which are  $CO_0, CO_1, \dots, CO_n$ .

A patent application program can be seen as a compound real option with the first (call) option as the patent application, with the corresponding strike price being the present value (as of  $T = 0$ ) of the capital outlays for the application program  $I_0$ , which is defined as follows:

$$I_0 = \text{Initial local application fee} + \frac{\text{PCT filing fee}}{(1+r)^{\frac{x_1}{12}}} + \frac{\text{Major costs required in national phase}}{(1+r)^{\frac{x_2}{12}}} \quad (7.4)$$

where  $r$  is the risk-free interest rate. The opportunity to invest in the follow-on commercialization project can be treated as the second (call) option with time to maturity of  $t$  years, whose underlying asset and exercise price is the present value (as of  $T = 0$ ) of the commercial project's expected future profits and the one-off investment of  $K$  respectively.

In the model, the cash inflows and outflows are given by four scenarios (taking into account the uncertainty concerning mainly the income in the second phase) and crisp numbers, respectively:

- Initial local application fee, PCT filing fee and the costs associated with internationalizing a patent application are fixed (there is no uncertainty involved) and thus will be represented as crisp numbers.
- Expected cash inflows from the patent exploitation will be represented as trapezoidal fuzzy numbers in the form of  $CI_j = (a_j, b_j, \alpha_j, \beta_j)$ .

Table 7.1: Cost and revenue cash-flow and present value of a patent with four scenarios in the commercialization phase

Years (starting from year t)	t	t+1	t+2	t+3	t+4	t+5	t+6	t+7	t+8
<b>Cash inflow</b>									
$a - \alpha$ (minimum possible)	1000	1300	1000	1800	1300	1600	1700	1400	2000
$a$ (minimum best guess)	2000	2300	2100	2300	1500	2000	1900	1600	2500
$b$ (maximum best guess)	2500	2900	2600	2800	1700	2600	2700	1900	3200
$b + \beta$ (maximum possible)	3500	3700	3200	3300	2000	3000	3300	2400	3500
<b>Present value of cash inflow</b>	opportunity cost of capital $k = 15\%$								
$a - \alpha$ (minimum possible)	1000	1130,43	756,14	1183,53	743,28	795,48	734,96	526,31	653,80
$a$ (minimum best guess)	2000	2000,00	1587,90	1512,29	857,63	994,35	821,42	601,50	817,25
$b$ (maximum best guess)	2500	2521,74	1965,97	1841,05	971,98	1292,66	1167,28	714,28	1046,09
$b + \beta$ (maximum possible)	3500	3217,39	2419,66	2169,80	1143,51	1491,53	1426,68	902,25	1144,16
<b>Cash outflows</b>	1200	1500	1500	2000	900	2000	2500	800	1200
<b>Present value of cash outflows</b>	risk-free rate $r = 4\%$								
	1200	1442,30	1386,83	1777,99	769,32	1643,85	1975,78	607,93	876,82
<b>Profit</b>									
$a - \alpha$	-200	-311,87	-630,69	-594,46	-26,04	-848,37	-1240,82	-81,62	-223,02
$a$	800	557,69	201,06	-265,70	88,30	-649,50	-1154,36	-6,43	-59,57
$b$	1300	1079,43	579,13	63,05	202,65	-351,19	-808,50	106,34	169,25
$b + \beta$	2300	1775,08	1032,82	391,81	374,18	-152,32	-549,10	294,31	267,32

- The representation of the outlays over the post-grant can be crisp, but for a more general description they may also be described in terms of fuzzy numbers when the uncertainty concerning the costs is significant. It is important to mention that a crisp number,  $c$ , can be seen as the trapezoidal fuzzy number  $\tilde{c} = (c, c, 0, 0)$ . Even if the value of the outlays is known precisely, they can still be included in the model as possibility distributions.

For simplicity, it is assumed that the patent is granted at year  $t = 3$  and is commercialized immediately. The values of the cash inflows and outflows (and their present values as of year  $t = 3$ ) during the 8-year commercialization project are listed in Table 7.1. The final outcome (the cumulative present value) at the end of the 8th year is a trapezoidal fuzzy number  $A = (a, b, \alpha, \beta) = (-488.51, 2340.19, 3668.41, 3393.93)$ . Using the formula for trapezoidal fuzzy numbers (in our case  $a < 0 < b$ ), the real option value can be calculated as  $ROV = 1101.83$ .

To calculate the value of the first option, the parameters are the following: initial local application fee is 100, the PCT filling fee is 200, the cost required in the national phase is 150,  $x_1$  is 12 months and  $x_2$  is 30 months. Using these numbers, one obtains that  $I_0 = 428.3$ . When we compare this to the present value of the ROV at the beginning of the application program (as of  $T = 0$ ), the result is

$$\max(ROV^* - I_0, 0) = 296.17,$$

which means that, for the company, it is profitable to file the patent application.

## 7.6 Discussion

There is reason to expect that the simplicity of the presented method is an advantage over more complex methods. Using triangular and trapezoidal fuzzy numbers make very easy implementations possible; this opens avenues for real option valuation to find its way to more practitioners. The method is flexible as it can be used when the fuzzy NPV is generated from scenarios or as fuzzy numbers from the beginning of the analysis.

As cash flows taking place in the future come closer, information changes, and uncertainty is reduced this should be reflected in the fuzzy NPV, the more there is uncertainty the wider the distribution should be, and when uncertainty is reduced the width of the distribution should decrease. Only under full certainty should the distribution be represented by a single number, as the method uses fuzzy NPV there is a possibility to have the size of the distribution decrease with a lesser degree of uncertainty, this is an advantage vis--vis probability based methods.

The common decision rules for ROV analysis are applicable with the ROV derived with the presented method. The single number NPV needed for comparison purposes can be derived from the (same) fuzzy NPV by calculating the fuzzy mean value. This means that in cases when all the values of the fuzzy NPV are greater than zero the single number NPV equals ROV, which indicates immediate investment.

## Chapter 8

# Conclusions and future research

Coping with uncertainty is a fundamental part of a decision making process. In real life problems it is hardly the case that every source of uncertainty can be reduced before taking action. The representation of incomplete information (deterministic uncertainty) has been a very active research field in the last decades and it will continue to grow due to the increasing importance of the topic. Fuzzy set theory is one of the developed theories to handle deterministic uncertainty. The research problems of the thesis concern a special type of fuzzy sets, namely fuzzy numbers.

When there are several uncertain variables present, an essential question is the modelling and interpretation of the interrelation between these variables. In the context of random variables, there exist well-established measures of interactivity to quantify the relationships between distributions. Correlation for fuzzy numbers was introduced in [13], but this measure is not always meaningful: when the level-sets of the joint distribution is not convex, the value of this coefficient can take its value outside the  $[-1, 1]$  interval. In Chapter 4 we have defined a novel measure of (relative) index of interactivity between marginal distributions  $A$  and  $B$  of a joint possibility distribution  $C$ .

This new index of interactivity is meaningful for any joint possibility distribution. This correlation coefficient is considered to be an index of interactivity between the  $\gamma$ -level sets of  $A$  and  $B$ . If  $[C]^\gamma$  is rectangular for  $0 \leq \gamma < 1$  then  $A$  and  $B$  are non-interactive and their index of interactivity is equal to zero. In the general case we have used the probabilistic correlation coefficient to measure the interactivity between the  $\gamma$ -level sets of  $A$  and  $B$ , which, loosely speaking, measures the 'strength' of concordant points as to the 'strength' of discordant points of  $[C]^\gamma$  with respect to the center of mass of  $[C]^\gamma$ . The use of this interactivity index was demonstrated through a series of examples with the most important joint distributions. This new concept can be applied in different areas of Operations Research and financial mathematics. One potential field can be the analysis of fuzzy time series where it can serve as a meaningful autocorrelation function.

In Chapter 5, we have introduced a correlation ratio for marginal possibility distributions of joint possibility distributions and illustrated this new principle by five examples.

This new concept has properties similar to the probabilistic correlation ratio. This measure can provide additional information concerning the interrelation of variables. One possible direction for applications is sensitivity analysis: if  $\mathbf{A}$  is a set of fuzzy numbers  $A_1, \dots, A_n$ , then  $\eta_f^2(G(\mathbf{A})|A_i)$  represents the fraction of the variance of  $G(\mathbf{A})$  which is "explained" by  $A_i$ . The main drawback of the correlation ratio is that there is an inherent problem with computing (or even estimating) it in a simple manner: the calculation of the conditional expectation  $E(G(\mathbf{A})|A_i)$  is not a straightforward task even in the simplest situations. If this problem of estimation can be solved in an effective way, the applications for example in portfolio estimation would be especially useful.

In Chapter 6, we have generalized the principles of possibilistic mean value, variance, covariance and correlation of fuzzy numbers to a more general class of fuzzy quantities: to quasi fuzzy numbers. We have shown some conditions for the existence of possibilistic mean value and variance for quasi fuzzy numbers. These results demonstrate that there is a wide class of quasi fuzzy numbers which can be considered in practical applications which require finite average properties but at the same time can include alternatives which are very unlikely to happen (although possible). These alternatives are usually overlooked in most of the real life applications which can result in disastrous consequences.

The methods of real option analysis and valuation in the recent years try to include decision makers already in the process of model construction by building the distributions on the basis of subjective judgements, as a result new models specifically built for this purpose have started to emerge. These models are not based on the strict assumptions of the classical financial option valuation models, but have adopted new views on the modeling of options and real options. The fuzzy pay-off method presented in Chapter 7 is one example of such models. It is based on cash-flow estimates from managers to form a simple pay-off distribution for the future that is then treated as a fuzzy number. From this fuzzy number the real option value can be calculated. Credibility theory is a construct that has been created to supplement the measurement of uncertainty and here it has been used in the context of real option valuation. The credibilistic expected value has been used for calculation and was compared to the possibilistic version of the model. Offering different choices in the derivation of the expected value is important. The simplicity of the presented methods can be an advantage in many situations over more complex real option valuation methods. Triangular and trapezoidal fuzzy numbers (represented by just three or four parameters) can be interpreted and processed in a very straightforward way; this opens avenues for real option valuation to more practitioners.

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**Part II**

**Original publications**





# The contribution of the author to the original publications

1. Main author. Wrote most of the paper (section 2 completely and most of section 3).
2. Joint author. Responsible for the mathematical analysis of the model and for the numerical examples (section 2).
3. Main author. Coordinated the writing of the paper, wrote most of the manuscript.
4. Joint author. Built the model together with the co-authors, wrote sections 4 and 5, and contributed to sections 2 and 3.
5. Joint author. Created the model together with the co-authors, wrote section 3.
6. Main author. Coordinated the writing of the paper, wrote most of the manuscript.



## **Paper 1**

Christer Carlsson, Robert Fullér and József Mezei, A Quantitative Approach to Quasi Fuzzy Numbers, IFSA World Congress, June 21-25, 2011, Surabaya, Indonesia, FN-001-1-FN-001-6.



# A Quantitative Approach to Quasi Fuzzy Numbers

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**Abstract:** In this paper we generalize the principles of possibilistic mean value, variance, covariance and correlation of fuzzy numbers to quasi fuzzy numbers. We will show some necessary and sufficient conditions for the existence of possibilistic mean value and variance for quasi fuzzy numbers. Considering the standard exponential probability distribution as a quasi fuzzy number we will compare the possibilistic and the probabilistic correlation coefficients.

## I. INTRODUCTION

A *fuzzy number*  $A$  is a fuzzy set in  $\mathbb{R}$  with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by  $\mathcal{F}$ . A *quasi fuzzy number*  $A$  is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions [2]

$$\lim_{t \rightarrow \infty} \mu_A(t) = 0, \quad \lim_{t \rightarrow -\infty} \mu_A(t) = 0.$$

A *quasi triangular fuzzy number* is a quasi fuzzy number with a unique maximizing point. Furthermore, we call  $\mathcal{Q}$  the family of all quasi fuzzy numbers. Quasi fuzzy numbers can also be considered as possibility distributions [6]. A  $\gamma$ -level set of a fuzzy set  $A$  in  $\mathbb{R}^m$  is defined by  $[A]^\gamma = \{x \in \mathbb{R}^m : \mu_A(x) \geq \gamma\}$ , if  $\gamma > 0$  and  $[A]^\gamma = \text{cl}\{x \in \mathbb{R}^m : \mu_A(x) > \gamma\}$  (the closure of the support of  $A$ ) if  $\gamma = 0$ .

If  $A$  is a fuzzy number, then  $[A]^\gamma$  is a closed convex (compact) subset of  $\mathbb{R}$  for all  $\gamma \in [0, 1]$ . If  $A$  is a quasi fuzzy number, then  $[A]^\gamma$  is a closed convex (compact) subset of  $\mathbb{R}$  for any  $\gamma > 0$ . Let us introduce the notations  $a_1(\gamma) = \min[A]^\gamma$ ,  $a_2(\gamma) = \max[A]^\gamma$ . In other words,  $a_1(\gamma)$  denotes the left-hand side and  $a_2(\gamma)$  denotes the right-hand side of the  $\gamma$ -cut, of  $A$  for any  $\gamma \in [0, 1]$ . A fuzzy set  $C$  in  $\mathbb{R}^2$  is said to be a joint possibility distribution of quasi fuzzy numbers  $A, B \in \mathcal{Q}$ , if it satisfies the relationships  $\max\{x \mid \mu_C(x, y)\} = \mu_B(y)$ ,  $\max\{y \mid \mu_C(x, y)\} = \mu_A(x)$ , for all  $x, y \in \mathbb{R}$ . Furthermore,  $A$  and  $B$  are called the marginal possibility distributions of  $C$ . A function  $f: [0, 1] \rightarrow \mathbb{R}$  is said to be a weighting function if  $f$  is non-negative, monoton increasing and satisfies the following normalization condition

$$\int_0^1 f(\gamma) d\gamma = 1.$$

Different weighting functions can give different (case-dependent) importances to  $\gamma$ -levels sets of quasi fuzzy numbers. It is motivated in part by the desire to give less importance to the lower levels of fuzzy sets [11] (it is why  $f$  should be monotone increasing).

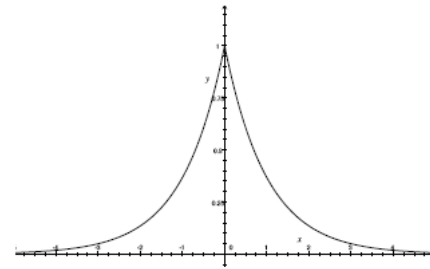


Fig. 1. A quasi triangular fuzzy number with membership function  $e^{-|x|}$ .

## II. POSSIBILISTIC MEAN VALUE, VARIANCE, COVARIANCE AND CORRELATION OF QUASI FUZZY NUMBERS

The possibilistic mean (or expected value), variance and covariance can be defined from the measure of possibilistic interactivity (as shown in [3], [9], [10]) but for simplicity, we will present the concept of possibilistic mean value, variance, covariance in a pure probabilistic setting. Let  $A \in \mathcal{F}$  be fuzzy number with  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$  and let  $U_\gamma$  denote a uniform probability distribution on  $[A]^\gamma$ ,  $\gamma \in [0, 1]$ . Recall that the probabilistic mean value of  $U_\gamma$  is equal to

$$M(U_\gamma) = \frac{a_1(\gamma) + a_2(\gamma)}{2},$$

and its probabilistic variance is computed by

$$\text{var}(U_\gamma) = \frac{(a_2(\gamma) - a_1(\gamma))^2}{12}.$$

The  $f$ -weighted *possibilistic mean value* (or expected value) of  $A \in \mathcal{F}$  is defined as [8]

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma,$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$  for all  $\gamma \in [0, 1]$ . If  $f(\gamma) = 1$  for all  $\gamma \in [0, 1]$  then we get

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma.$$

That is,  $f(\gamma) \equiv 1$  the  $f$ -weighted possibilistic mean value coincides with the (i) generative expectation of fuzzy numbers introduced by Chanas and Nowakowski in ([4], page 47); (ii) middle-point-of-the-mean-interval defuzzication method proposed by Yager in ([15], page 161). In this paper we will use the *natural* weighting function  $f(\gamma) = 2\gamma$ . In this case the possibilistic mean value is, denoted by  $E(A)$ , defined by,

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma \\ &= \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) d\gamma, \end{aligned} \quad (1)$$

which the possibilistic mean value of  $A$  originally introduced by Carlsson and Fullér in 2001 [1]. We note here that from the equality

$$\begin{aligned} E(A) &= \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) d\gamma \\ &= \frac{\int_0^1 \gamma \cdot \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma}{\int_0^1 \gamma d\gamma}, \end{aligned}$$

it follows that  $E(A)$  is nothing else but the level-weighted average of the arithmetic means of all  $\gamma$ -level sets, that is, the weight of the arithmetic mean of  $a_1(\gamma)$  and  $a_2(\gamma)$  is just  $\gamma$ .

**Note 1.** There exist several other ways to define mean values of fuzzy numbers, e.g. Dubois and Prade [5] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. Using evaluation measures, Yoshida et al [16] introduced a possibility mean, a necessity mean and a credibility mean of fuzzy numbers that are different from (1). Surveying the results in quantitative possibility theory, Dubois [7] showed that some notions (e.g. cumulative distributions, mean values) in statistics can naturally be interpreted in the language of possibility theory.

Now we will extend the concept of possibilistic mean value to the family of quasi fuzzy numbers.

**Definition II.1.** The  $f$ -weighted possibilistic mean value of  $A \in \mathcal{Q}$  is defined as

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma,$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$  for all  $\gamma > 0$ . The value of  $E_f(A)$  does not depend on the boundedness of the support of  $A$ .

The possibilistic mean value is originally defined for fuzzy numbers (i.e. quasi fuzzy numbers with bounded support). If the support of a quasi fuzzy number  $A$  is unbounded then its possibilistic mean value might even not exist. However, for a symmetric quasi fuzzy number  $A$  we get  $E_f(A) = a$ , where  $a$  is the center of symmetry, for any weighting function  $f$ .

Now we will characterize the family of quasi fuzzy numbers for which it is possible to calculate the possibilistic mean value. First we show an example for a quasi triangular fuzzy number that does not have a mean value.

**Example II.1.** Consider the following quasi triangular fuzzy number

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{\sqrt{x+1}} & \text{if } 0 \leq x \end{cases}$$

In this case

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma^2} - 1,$$

and its possibilistic mean value can not be computed, since the following integral does not exist (not finite),

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma \\ &= \int_0^1 \left( \frac{1}{\gamma^2} - 1 \right) \gamma d\gamma = \int_0^1 \left( \frac{1}{\gamma} - \gamma \right) d\gamma. \end{aligned}$$

**Note 2.** This example is very important: if the membership function of the quasi fuzzy number tends to zero slower than the function  $1/\sqrt{x}$  then it is not possible to calculate the possibilistic mean value, (clearly, the value of the integral will be infinitive), otherwise the possibilistic mean value does exist.

To show this, suppose that there exists  $\varepsilon > 0$ , such that the membership function of quasi fuzzy number  $A$  satisfies the property,

$$\mu_A(x) = O(x^{-\frac{1}{2}-\varepsilon})$$

if  $x \rightarrow +\infty$ . This means that there exists and  $x_0 \in \mathbb{R}$  such that,

$$\mu_A(x) \leq Mx^{-\frac{1}{2}-\varepsilon},$$

if  $x > x_0$  and where  $M$  is a positive real number. So the possibilistic mean value of  $A$  is bonded from above by

$$M^{-\frac{1}{-\frac{1}{2}-\varepsilon}}$$

multiplied by the possibilistic mean value of a quasi fuzzy number with membership function  $x^{-\frac{1}{2}-\varepsilon}$  plus an additional constant (because of the properties of a quasi fuzzy number we know that the interval  $[0, x_0]$  accounts for a finite value in the integral).

Suppose that,

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \\ x^{-\frac{1}{2}-\varepsilon} & \text{if } x \geq 1 \end{cases}$$

A similar reasoning holds for negative fuzzy numbers with membership function  $(-x)^{-\frac{1}{2}-\varepsilon}$ . Then we get,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \gamma^{-\frac{1}{\frac{1}{2}-\varepsilon}},$$

and since

$$\frac{\varepsilon - \frac{1}{2}}{\varepsilon + \frac{1}{2}} \neq 1,$$

we can calculate the possibilistic mean value of  $A$  as,

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 \gamma^{-\frac{1}{\varepsilon+\frac{1}{2}}} \gamma d\gamma \\ &= \int_0^1 \gamma^{-\frac{\varepsilon-\frac{1}{2}}{\varepsilon+\frac{1}{2}}} d\gamma = \left(\varepsilon + \frac{1}{2}\right) \left[\gamma^{\frac{1}{\varepsilon+\frac{1}{2}}}\right]_0^1 = \varepsilon + 1/2 \end{aligned}$$

**Theorem II.1.** If  $A$  is a non-symmetric quasi fuzzy number then  $E_f(A)$  exists if and only if there exist real numbers  $\varepsilon, \delta > 0$ , such that,

$$\mu_A(x) = O(x^{-\frac{1}{2}-\varepsilon}),$$

if  $x \rightarrow +\infty$  and

$$\mu_A(x) = O((-x)^{-\frac{1}{2}-\delta}),$$

if  $x \rightarrow -\infty$ .

**Note 3.** If we consider other weighting functions, we need to require that  $\mu_A(x) = O(x^{-1-\varepsilon})$ , when  $x \rightarrow +\infty$  (in the worst case, when  $f(\gamma) = 1$ ,  $\frac{1}{\gamma}$  is the critical growth rate.)

**Example II.2.** Consider the following quasi triangular fuzzy number,

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{x+1} & \text{if } 1 \leq x \end{cases}$$

In this case we have,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma} - 1,$$

and its possibilistic mean value is,

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 \left(\frac{1}{\gamma} - 1\right) \gamma d\gamma \\ &= \int_0^1 (1 - \gamma) d\gamma = 1/2. \end{aligned}$$

This example is very important since the volume of  $A$  can not be normalized since  $\int_0^\infty \mu_A(x) dx$  does not exist. In other words,  $\mu_A$  can not be considered as a density function of any random variable.

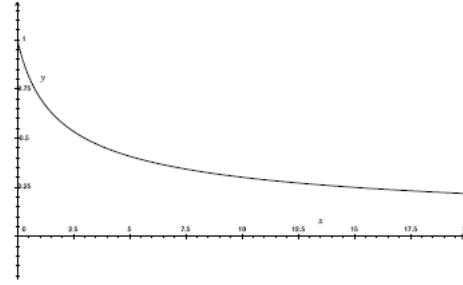


Fig. 2. Quasi triangular fuzzy number  $1/(x+1)$ ,  $x \geq 0$ .

The measure of  $f$ -weighted possibilistic variance of a fuzzy number  $A$  is the  $f$ -weighted average of the probabilistic variances of the respective uniform distributions on the level sets of  $A$ . That is, the  $f$ -weighted possibilistic variance of  $A$  is defined by [9]

$$\begin{aligned} \text{Var}_f(A) &= \int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma \\ &= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma. \end{aligned}$$

Now we will extend the concept of possibilistic variance to the family of quasi fuzzy numbers.

**Definition II.2.** The measure of  $f$ -weighted possibilistic variance of a quasi fuzzy number  $A$  is the  $f$ -weighted average of the probabilistic variances of the respective uniform distributions on the level sets of  $A$ . That is, the  $f$ -weighted possibilistic variance of  $A$  is defined by

$$\begin{aligned} \text{Var}_f(A) &= \int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma \\ &= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma. \end{aligned}$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$  for all  $\gamma > 0$ . The value of  $\text{Var}_f(A)$  does not depend on the boundedness of the support of  $A$ . If  $f(\gamma) = 2\gamma$  then we simple write  $\text{Var}(A)$ .

From the definition it follows that in this case we can not make any distinction between the symmetric and non-symmetric case. And it is also obvious, since in the definition we have the square of the  $a_1(\gamma)$  and  $a_2(\gamma)$  functions, that the decreasing rate of the membership function has to be the square of the mean value case. We can conclude:

**Theorem II.2.** If  $A$  is a quasi fuzzy number then  $\text{Var}(A)$  exists if and only if there exist real numbers  $\varepsilon, \delta > 0$ , such

$$\mu_A(x) = O(x^{-1-\varepsilon})$$

if  $x \rightarrow +\infty$  and

$$\mu_A(x) = O((-x)^{-1-\delta}),$$

if  $x \rightarrow -\infty$ .

**Note 4.** If we consider other weighting functions, we need to require that

$$\mu_A(x) = O(x^{-2-\varepsilon}),$$

when  $x \rightarrow +\infty$  (in the worst case, when  $f(\gamma) = 1$ ,  $\frac{1}{\sqrt{\gamma}}$  is the critical growth rate.)

**Example II.3.** Consider again the quasi triangular fuzzy number;

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{x+1} & \text{if } 1 \leq x \end{cases}$$

In this case we have,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = \frac{1}{\gamma} - 1,$$

and its possibilistic variance does not exist since

$$\int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} 2\gamma d\gamma = \int_0^1 \frac{(1/\gamma - 1)^2}{12} 2\gamma d\gamma = \infty.$$

In 2004 Fullér and Majlender [9] introduced a measure of possibilistic covariance between marginal distributions of a joint possibility distribution  $C$  as the expected value of the interactivity relation between the  $\gamma$ -level sets of its marginal distributions. In 2005 Carlsson, Fullér and Majlender [3] showed that the possibilistic covariance between fuzzy numbers  $A$  and  $B$  can be written as the weighted average of the probabilistic covariances between random variables with uniform joint distribution on the level sets of their joint possibility distribution  $C$ . The  $f$ -weighted measure of possibilistic covariance between  $A, B \in \mathcal{F}$ , (with respect to their joint distribution  $C$ ), defined by [9], can be written as

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\text{cov}(X_\gamma, Y_\gamma)$  denotes their probabilistic covariance.

Now we will extend the concept of possibilistic covariance to the family of quasi fuzzy numbers.

**Definition II.3.** The  $f$ -weighted measure of possibilistic covariance between  $A, B \in \mathcal{Q}$ , (with respect to their joint distribution  $C$ ), is defined by,

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for any  $\gamma > 0$ .

It is easy to see that the possibilistic covariance is an absolute measure in the sense that it can take any value from the real line. To have a relative measure of interactivity between marginal distributions Fullér, Mezei and Várlaki

introduced the normalized covariance in 2010 (see [10]). A normalized  $f$ -weighted index of interactivity of  $A, B \in \mathcal{F}$  (with respect to their joint distribution  $C$ ) is defined by

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}}$$

and, where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ .

In other words, the ( $f$ -weighted) index of interactivity is nothing else, but the  $f$ -weighted average of the probabilistic correlation coefficients  $\rho(X_\gamma, Y_\gamma)$  for all  $\gamma \in [0, 1]$ . It is clear that for any joint possibility distribution this correlation coefficient always takes its value from interval  $[-1, 1]$ , since  $\rho(X_\gamma, Y_\gamma) \in [-1, 1]$  for any  $\gamma \in [0, 1]$  and  $\int_0^1 f(\gamma) d\gamma = 1$ . Since  $\rho_f(A, B)$  measures an average index of interactivity between the level sets of  $A$  and  $B$ , we may call this measure as the  $f$ -weighted possibilistic correlation coefficient.

Now we will extend the concept of possibilistic correlation to the family of quasi fuzzy numbers.

**Definition II.4.** The  $f$ -weighted possibilistic correlation coefficient of  $A, B \in \mathcal{Q}$  (with respect to their joint distribution  $C$ ) is defined by

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}}$$

and, where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for any  $\gamma > 0$ .

### III. PROBABILITY VERSUS POSSIBILITY: THE CASE OF EXPONENTIAL FUNCTION

Now we will calculate the possibilistic mean value and variance of a quasi triangular fuzzy number defined by the membership function  $e^{-x}$ ,  $x \geq 0$ , which can also be seen as a density function of a standard exponential random variable. In probability theory and statistics, the exponential distribution is a family of continuous probability distributions. It describes the time between events in a *Poisson process*, i.e. a process in which events occur continuously and independently at a constant average rate.

Consider the following quasi triangular fuzzy number

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$$

From  $\int_0^\infty \mu_A(x) dx = 1$  it follows that  $\mu_A$  can also be considered as the density function of a standard exponential random variable (with parameter one). It is well-known



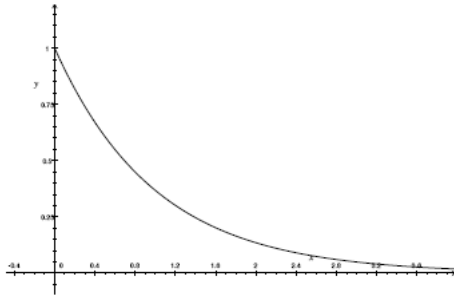


Fig. 3. Quasi triangular fuzzy number and density function of an exponential random variable with parameter one:  $e^{-x}$ ,  $x \geq 0$ .

that the mean value and the variance of this probability distribution is equal to one. In the fuzzy case we have,

$$a_1(\gamma) = 0, \quad a_2(\gamma) = -\ln \gamma,$$

and its possibilistic mean value is

$$E(A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma d\gamma = \int_0^1 -(\ln \gamma)\gamma d\gamma = \frac{1}{4},$$

and its possibilistic variance is,

$$\begin{aligned} \text{Var}(A) &= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} 2\gamma d\gamma \\ &= \int_0^1 \frac{(-\ln \gamma)^2}{6} \gamma d\gamma = \frac{1}{24}. \end{aligned}$$

Let  $C$  be the joint possibility distribution, defined by the membership function,

$$\mu_C(x, y) = e^{-(x+y)}, \quad x \geq 0, y \geq 0,$$

of quasi fuzzy numbers  $A$  and  $B$  with membership functions

$$\mu_A(x) = e^{-x}, \quad x \geq 0, \quad \text{and} \quad \mu_B(y) = e^{-y}, \quad y \geq 0.$$

In other words, the membership function of  $C$  is defined by a simple multiplication (by Larsen t-norm [13]) of the membership values of  $\mu_A(x)$  and  $\mu_B(y)$ , that is,  $\mu_C(x, y) = \mu_A(x) \times \mu_B(y)$ . The  $\gamma$ -cut of  $C$  can be computed by

$$[C]^\gamma = \{(x, y) \mid x + y \leq -\ln \gamma; x, y \geq 0\}.$$

Then

$$\begin{aligned} M(X_\gamma) &= M(Y_\gamma) = -\frac{\ln \gamma}{3}, \\ M(X_\gamma^2) &= M(Y_\gamma^2) = \frac{(\ln \gamma)^2}{6}, \end{aligned}$$

and,

$$\begin{aligned} \text{var}(X_\gamma) &= M(X_\gamma^2) - M(X_\gamma)^2 \\ &= \frac{(\ln \gamma)^2}{6} - \frac{(\ln \gamma)^2}{9} \\ &= \frac{(\ln \gamma)^2}{18}. \end{aligned}$$

Similarly we obtain,

$$\text{var}(Y_\gamma) = \frac{(\ln \gamma)^2}{18}.$$

Furthermore,

$$M(X_\gamma Y_\gamma) = \frac{(\ln \gamma)^2}{12},$$

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(\ln \gamma)^2}{36},$$

we can calculate the probabilistic correlation by

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

That is,  $\rho(X_\gamma, Y_\gamma) = -1/2$  for any  $\gamma > 0$ . Consequently, their possibilistic correlation coefficient is,

$$\rho_f(A, B) = -1/2$$

for any weighting function  $f$ .

On the other hand, in a probabilistic context,  $\mu_C(x, y) = \mu_A(x) \times \mu_B(y) = e^{-(x+y)}$  can be also considered as the joint density function of independent exponential marginal probability distributions with parameter one. That is, in a probabilistic context, their (probabilistic) correlation coefficient is equal to zero.

**Note 5.** The probabilistic correlation coefficient between two standard exponential marginal probability distributions can not go below  $(1 - \pi^2/6)$ . Really, the lower limit, denoted by  $\tau$ , can be computed from,

$$\begin{aligned} &\int_0^\infty \int_0^\infty (1 - e^{-x} - e^{-y})^+ - (1 - e^{-x})(1 - e^{-y}) dx dy \\ &= - \int_0^\infty \int_0^\infty e^{-x} e^{-y} dx dy \\ &\quad + \iint_{0 < x, 0 < y, 1 < e^{-x} + e^{-y}} (1 - e^{-x} - e^{-y})^+ dx dy \\ &= -1 + \iint_{0 < x, 0 < y, 1 < e^{-x} + e^{-y}} (2e^{-x} - 1) dx dy = \tau \end{aligned}$$

using the substitutions  $u = e^{-x}$ ,  $v = e^{-y}$  we get,

$$\begin{aligned}\tau &= -1 + \iint_{u < 1, v < 1, u+v > 1} \left( \frac{2}{u} - \frac{1}{uv} \right) dudv \\ &= -1 + \int_0^1 \frac{1}{u} \int_0^1 \left( 2 - \frac{1}{v} \right) dv du \\ &= 1 + \int_0^1 \frac{2u + \log(1-u)}{u} du \\ &= -1 + \int_0^1 \frac{\log(1-u)}{u} du \\ &= \int_0^1 \sum_{k=1}^{\infty} \frac{u^{k-1}}{k} du \\ &= 1 - \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= 1 - \frac{\pi^2}{6}.\end{aligned}$$

In the case of possibility distributions there is no known lower limit [12].

If the joint possibility distribution  $C$  is given by the minimum operator (Mamdani t-norm [14]),

$$\mu_C(x, y) = \min\{\mu_A(x), \mu_B(y)\} = \min\{e^{-x}, e^{-y}\},$$

$x \geq 0$ ,  $y \geq 0$ , then  $A$  and  $B$  are non-interactive marginal possibility distributions and, therefore, their possibilistic correlation coefficient equal to zero.

## IV. SUMMARY

We have generalized the principles of possibilistic mean value, variance, covariance and correlation of fuzzy numbers to a more general class of fuzzy quantities: to quasi fuzzy numbers. We have shown some necessary and sufficient conditions for the existence of possibilistic mean value and variance for quasi fuzzy numbers.

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## **Paper 2**

Mikael Collan, Robert Fullér and József Mezei, A Fuzzy Pay-off Method for Real Option Valuation, *Journal of Applied Mathematics and Decision Sciences*, 2009(2009), 1-14.



## Research Article

# A Fuzzy Pay-Off Method for Real Option Valuation

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Real option analysis offers interesting insights on the value of assets and on the profitability of investments, which has made real options a growing field of academic research and practical application. Real option valuation is, however, often found to be difficult to understand and to implement due to the quite complex mathematics involved. Recent advances in modeling and analysis methods have made real option valuation easier to understand and to implement. This paper presents a new method (fuzzy pay-off method) for real option valuation using fuzzy numbers that is based on findings from earlier real option valuation methods and from fuzzy real option valuation. The method is intuitive to understand and far less complicated than any previous real option valuation model to date. The paper also presents the use of number of different types of fuzzy numbers with the method and an application of the new method in an industry setting.

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## 1. Introduction

Real option valuation is based on the observation that the possibilities financial options give their holder resemble the possibilities to invest in real investments and possibilities found within real investments, that is, managerial flexibility: “an irreversible investment opportunity is much like a financial call option” [1]. In other words, real option valuation is treating investment opportunities and the different types of managerial flexibility as options and valuing them with option valuation models. Real options are useful both, as a mental model for strategic and operational decision-making, and as a valuation and numerical analysis tool. This paper concentrates on the use of real options in numerical analysis, and particularly on the derivation of the real option value for a given investment opportunity, or identified managerial flexibility.

Real options are commonly valued with the same methods that have been used to value financial options, that is, with Black-Scholes option pricing formula [2], with the

binomial option valuation method [3], with Monte-Carlo-based methods [4], and with a number of later methods based on these. Most of the methods are complex and demand a good understanding of the underlying mathematics, issues that make their use difficult in practice. In addition these models are based on the assumption that they can quite accurately mimic the underlying markets as a process, an assumption that may hold for some quite efficiently traded financial securities, but may not hold for real investments that do not have existing markets or have markets that can by no means be said to exhibit even weak market efficiency.

Recently, a novel approach to real option valuation, called the Datar-Mathews method (DMM) was presented in [5–7], where the real option value is calculated from a pay-off distribution, derived from a probability distribution of the net present value (NPV) for a project that is generated with a (Monte-Carlo) simulation. The authors show that the results from the method converge to the results from the analytical Black-Scholes method. The method presented greatly simplifies the calculation of the real option value, making it more transparent and brings real option valuation as a method a big leap closer to practitioners. The most positive issue in the DMM is that it does not suffer from the problems associated with the assumptions connected to the market processes connected to the Black-Scholes and the binomial option valuation methods. The DMM utilizes cash-flow scenarios as an input to a Monte Carlo simulation to derive a distribution for the future investment outcomes. This distribution is then used to create a pay-off distribution for the investment. The DMM is highly compatible with the way cash-flow-based profitability analysis is commonly done in companies, because it can use the same type of inputs as NPV analysis.

All of the afore-mentioned models and methods use probability theory in their treatment of uncertainty, there are, however, other ways than probability to treat uncertainty, or imprecision in future estimates, namely, fuzzy logic and fuzzy sets. In classical set theory an element either (fully) belongs to a set or does not belong to a set at all. This type of bivalence, or true/false, logic is commonly used in financial applications (and is a basic assumption of probability theory). Bivalence logic, however, presents a problem, because financial decisions are generally made under uncertainty. Uncertainty in the financial investment context means that it is in practice impossible, *ex ante* to give absolutely correct precise estimates of, for example, future cash-flows. There may be a number of reasons for this, see, for example, [8], however, the bottom line is that our estimations about the future are imprecise.

Fuzzy sets are sets that allow (have) gradation of belonging, such as “a future cash flow at year ten is about  $x$  euro”. This means that fuzzy sets can be used to formalize inaccuracy that exists in human decision making and as a representation of vague, uncertain, or imprecise knowledge, for example, future cash-flow estimation, which human reasoning is especially adaptive to. “Fuzzy set-based methodologies blur the traditional line between qualitative and quantitative analysis, since the modeling may reflect more the type of information that is available rather than researchers’ preferences” [9], and indeed in economics “the use of fuzzy subsets theory leads to results that could not be obtained by classical methods” [10]. The origins of fuzzy sets date back to an article by Lotfi Zadeh [11] where he developed an algebra for what he called fuzzy sets. This algebra was created to handle imprecise elements in our decision-making processes, and is the formal body of theory that allows the treatment of practically all decisions in an uncertain environment. “Informally, a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not” [12].

In the following subsection we will shortly present fuzzy sets and fuzzy numbers and continue shortly on using fuzzy numbers in option valuation. We will then present a

new method for valuation of real options from fuzzy numbers that is based on the previous literature on real option valuation, especially the findings presented in [5] and on fuzzy real option valuation methods, we continue by illustrating the use of the method with a selection of different types of fuzzy numbers and with a case application of the new method in an industry setting, and close with a discussion and conclusions.

### 1.1. Fuzzy Sets and Fuzzy Numbers

A *fuzzy subset*  $A$  of a nonempty  $X$  set can be defined as a set of ordered pairs, each with the first element from  $X$ , and the second element from the interval  $[0, 1]$ , with exactly one-ordered pair presents for each element of  $X$ . This defines a mapping,

$$\mu_A : A \rightarrow [0, 1], \quad (1.1)$$

between elements of the set  $X$  and values in the interval  $[0, 1]$ . The value zero is used to represent complete nonmembership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership. The set  $X$  is referred to as the universe of discourse for the fuzzy subset  $A$ . Frequently, the mapping  $\mu_A$  is described as a function, the membership function of  $A$ . The degree to which the statement  $x$  is in  $A$  is true is determined by finding the ordered pair  $(x, \mu_A(x))$ . The degree of truth of the statement is the second element of the ordered pair. It is clear that  $A$  is completely determined by the set of tuples

$$A = \{(x, \mu_A(x)) \mid x \in X\}. \quad (1.2)$$

It should be noted that the terms *membership function* and *fuzzy subset* get used interchangeably and frequently we will write simply  $A(x)$  instead of  $\mu_A(x)$ . A  $\gamma$ -level set (or  $\gamma$ -cut) of a fuzzy set  $A$  of  $X$  is a nonfuzzy set denoted by  $[A]^\gamma$  and defined by

$$[A]^\gamma = \{t \in X \mid A(t) \geq \gamma\}, \quad (1.3)$$

if  $\gamma > 0$  and  $\text{cl}(\text{supp } A)$  if  $\gamma = 0$ , where  $\text{cl}(\text{supp } A)$  denotes the closure of the support of  $A$ . A fuzzy set  $A$  of  $X$  is called *convex* if  $[A]^\gamma$  is a convex subset of  $X$  for all  $\gamma \in [0, 1]$ . A fuzzy number  $A$  is a fuzzy set of the real line with a normal, (fuzzy) convex, and continuous membership function of bounded support [13]. Fuzzy numbers can be considered as possibility distributions.

*Definition 1.1.* Let  $A$  be a fuzzy number. Then  $[A]^\gamma$  is a closed convex (compact) subset of  $\mathbb{R}$  for all  $\gamma \in [0, 1]$ . Let us introduce the notations

$$a_1(\gamma) = \min[A]^\gamma, \quad a_2(\gamma) = \max[A]^\gamma \quad (1.4)$$

In other words,  $a_1(\gamma)$  denotes the left-hand side and  $a_2(\gamma)$  denotes the right-hand side of the  $\gamma$ -cut,  $\gamma \in [0, 1]$ .

*Definition 1.2.* A fuzzy set  $A$  is called triangular fuzzy number with peak (or center)  $a$ , left width  $\alpha > 0$  and right width  $\beta > 0$  if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a, \\ 1 - \frac{t-a}{\beta} & \text{if } a \leq t \leq a + \beta, \\ 0 & \text{otherwise,} \end{cases} \quad (1.5)$$

and we use the notation  $A = (a, \alpha, \beta)$ . It can easily be verified that

$$[A]^\gamma = [a - (1 - \gamma)\alpha, a + (1 - \gamma)\beta], \quad \forall \gamma \in [0, 1]. \quad (1.6)$$

The support of  $A$  is  $(a - \alpha, a + \beta)$ . A triangular fuzzy number with center  $a$  may be seen as a fuzzy quantity “ $x$  is approximately equal to  $a$ ”.

*Definition 1.3.* The *possibilistic (or fuzzy) mean value* of fuzzy number  $A$  with  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$  is defined in [13] by

$$\begin{aligned} E(A) &= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} 2\gamma \, d\gamma \\ &= \int_0^1 (a_1(\gamma) + a_2(\gamma)) \gamma \, d\gamma. \end{aligned} \quad (1.7)$$

*Definition 1.4.* A fuzzy set  $A$  is called trapezoidal fuzzy number with tolerance interval  $[a, b]$ , left width  $\alpha$ , and right width  $\beta$  if its membership function has the following form:

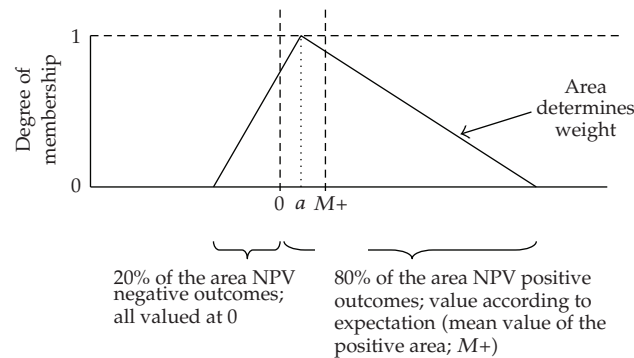
$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a, \\ 1 & \text{if } a \leq t \leq b, \\ 1 - \frac{t-b}{\beta} & \text{if } b \leq t \leq b + \beta, \\ 0 & \text{otherwise,} \end{cases} \quad (1.8)$$

and we use the notation

$$A = (a, b, \alpha, \beta). \quad (1.9)$$

It can easily be shown that  $[A]^\gamma = [a - (1 - \gamma)\alpha, b + (1 - \gamma)\beta]$  for all  $\gamma \in [0, 1]$ . The support of  $A$  is  $(a - \alpha, b + \beta)$ .





**Figure 1:** A triangular fuzzy number  $A$ , defined by three points  $\{a, \alpha, \beta\}$  describing the NPV of a prospective project; (percentages 20% and 80% are for illustration purposes only).

Fuzzy set theory uses fuzzy numbers to quantify subjective fuzzy observations or estimates. Such subjective observations or estimates can be, for example, estimates of future cash flows from an investment. To estimate future cash flows and discount rates “one usually employs educated guesses, based on expected values or other statistical techniques” [14], which is consistent with the use of fuzzy numbers. In practical applications the most used fuzzy numbers are trapezoidal and triangular fuzzy numbers. They are used because they make many operations possible and are intuitively understandable and interpretable.

When we replace nonfuzzy numbers (crisp, single) numbers that are commonly used in financial models with fuzzy numbers, we can construct models that include the inaccuracy of human perception, or ability to forecast, within the (fuzzy) numbers. This makes these models more in line with reality, as they do not simplify uncertain distribution-like observations to a single-point estimate that conveys the sensation of no-uncertainty. Replacing nonfuzzy numbers with fuzzy numbers means that the models that are built must also follow the rules of fuzzy arithmetic.

## 1.2. Fuzzy Numbers in Option Valuation

Fuzzy numbers (fuzzy logic) have been adopted to option valuation models in (binomial) pricing an option with a fuzzy pay-off, for example, in [15], and in Black-Scholes valuation of financial options in, for example, [16]. There are also some option valuation models that present a combination of probability theory and fuzzy sets, for example, [17]. Fuzzy numbers have also been applied to the valuation of real options in, for example, [18–20]. More recently there are a number of papers that present the application of fuzzy real option models in the industry setting, for example, [21, 22]. There are also specific fuzzy models for the analysis of the value of optionality for very large industrial real investments, for example, [23].

## 2. New Fuzzy Pay-Off Method for Valuation of Real Options from Fuzzy Numbers

Two recent papers [5, 6] present a practical probability theory-based Datar-Mathews method for the calculation of real option value and show that the method and results from the method

are mathematically equivalent to the Black-Sholes formula [2]. The method is based on simulation-generated probability distributions for the NPV of future project outcomes. The project outcome probability distributions are used to generate a pay-off distribution, where the negative outcomes (subject to terminating the project) are truncated into one chunk that will cause a zero pay-off, and where the probability-weighted average value of the resulting pay-off distribution is the real option value. The DMM shows that the real-option value can be understood as the probability-weighted average of the pay-off distribution. We use fuzzy numbers in representing the expected future distribution of possible project costs and revenues, and hence also the profitability (NPV) outcomes. The fuzzy NPV, a fuzzy number, is the pay-off distribution from the project.

The method presented in [5] implies that the weighted average of the positive outcomes of the pay-off distribution is the real option value; in the case with fuzzy numbers the weighted average is the fuzzy mean value of the positive NPV outcomes. Derivation of the fuzzy mean value is presented in [13]. This means that calculating the ROV from a fuzzy NPV (distribution) is straightforward, it is the fuzzy mean of the possibility distribution with values below zero counted as zero, that is, the area-weighted average of the fuzzy mean of the positive values of the distribution and zero (for negative values)

*Definition 2.1.* We calculate the real option value from the fuzzy NPV as follows:

$$\text{ROV} = \frac{\int_0^{\infty} A(x)dx}{\int_{-\infty}^{\infty} A(x)dx} \times E(A_+), \quad (2.1)$$

where  $A$  stands for the fuzzy NPV,  $E(A_+)$  denotes the fuzzy mean value of the positive side of the NPV, and  $\int_{-\infty}^{\infty} A(x)dx$  computes the area below the whole fuzzy number  $A$  while  $\int_0^{\infty} A(x)dx$  computes the area below the positive part of  $A$ .

It is easy to see that when the whole fuzzy number is above zero, then ROV is the fuzzy mean of the fuzzy number, and when the whole fuzzy number is below zero, the ROV is zero.

The components of the new method are simply the observation that real option value is the probability-weighted average of the positive values of a pay-off distribution of a project, which is the fuzzy NPV of the project, and that for fuzzy numbers, the probability-weighted average of the positive values of the pay-off distribution is the weighted fuzzy mean of the positive values of the fuzzy NPV, when we use fuzzy numbers.

### ***2.1. Calculating the ROV with the Fuzzy Pay-Off Method with a Selection of Different Types of Fuzzy Numbers***

As the form of a fuzzy number may vary, the most used forms are the triangular and trapezoidal fuzzy numbers. These are very usable forms, as they are easy to understand and can be simply defined by three (triangular) and four (trapezoidal) values.

We should calculate the positive area and the fuzzy mean of the positive area of a triangular fuzzy pay-off  $A = (a, \alpha, \beta)$  in the case of  $a - \alpha < 0 < a$ . Variable  $z$ , where  $0 \leq z \leq \alpha$ , represents the distance of a general cut point from  $a - \alpha$  at which we separate the triangular fuzzy number (distribution) into two parts—for our purposes the variable  $z$  gets the value

$\alpha - a$  (we are interested in the positive part of  $A$ ). Let us introduce the notation

$$(A | z)(t) = \begin{cases} 0 & \text{if } t \leq a - \alpha + z, \\ A(t) & \text{otherwise,} \end{cases} \quad (2.2)$$

for the membership function of the right-hand side of a triangular fuzzy number truncated at point  $a - \alpha + z$ , where  $0 \leq z \leq \alpha$ .

Then we can compute the expected value of this truncated triangular fuzzy number:

$$E(A | z) = I_1 + I_2 = \int_0^{z_1} \gamma(a - \alpha + z + a + (1 - \gamma)\beta) d\gamma + \int_{z_1}^1 \gamma(a - (1 - \gamma)\alpha + a + (1 - \gamma)\beta) d\gamma, \quad (2.3)$$

where

$$z_1 = 1 - \frac{\alpha - z}{\alpha} = \frac{z}{\alpha}, \quad (2.4)$$

and the integrals are computed by

$$\begin{aligned} I_1 &= \int_0^{z_1} [(2a - \alpha + z + \beta)\gamma - \beta\gamma^2] d\gamma \\ &= (2a - \alpha + z + \beta) \frac{z^2}{2\alpha^2} - \beta \frac{z^3}{3\alpha^3}, \\ I_2 &= \int_{z_1}^1 [(2a + \beta - \alpha)\gamma - \gamma^2(\beta - \alpha)] d\gamma \\ &= (2a + \beta - \alpha) \left( \frac{1}{2} - \frac{z^2}{2\alpha^2} \right) - (\beta - \alpha) \left( \frac{1}{3} - \frac{z^3}{3\alpha^3} \right), \end{aligned} \quad (2.5)$$

that is,

$$\begin{aligned} I_1 + I_2 &= (2a - \alpha + z + \beta) \times \frac{z^2}{2\alpha^2} - \beta \times \frac{z^3}{3\alpha^3} + (2a + \beta - \alpha) \times \left( \frac{1}{2} - \frac{z^2}{2\alpha^2} \right) \\ &\quad - (\beta - \alpha) \times \left( \frac{1}{3} - \frac{z^3}{3\alpha^3} \right) = \frac{z^3}{2\alpha^2} + \frac{2a - \alpha + \beta}{2} + \frac{\alpha - \beta}{3} - \alpha \times \frac{z^3}{3\alpha^3}, \end{aligned} \quad (2.6)$$

and we get,

$$E(A | z) = \frac{z^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}. \quad (2.7)$$

If  $z = \alpha - a$ , then  $A | z$  becomes  $A_+$ , the positive side of  $A$ , and therefore, we get

$$E(A_+) = \frac{(\alpha - a)^3}{6\alpha^2} + a + \frac{\beta - \alpha}{6}. \quad (2.8)$$

To compute the real option value with the afore-mentioned formulas we must calculate the ratio between the positive area of the triangular fuzzy number and the total area of the same number and multiply this by  $E(A_+)$ , the fuzzy mean value of the positive part of the fuzzy number  $A$ , according to (2.1).

For computing the real option value from an NPV (pay-off) distribution of a trapezoidal form we must consider a trapezoidal fuzzy pay-off distribution  $A$  defined by

$$A(u) = \begin{cases} \frac{u}{\alpha} - \frac{a_1 - \alpha}{\alpha} & \text{if } a_1 - \alpha \leq u \leq a_1, \\ 1 & \text{if } a_1 \leq u \leq a_2, \\ \frac{u}{-\beta} + \frac{a_2 + \beta}{\beta} & \text{if } a_2 \leq u \leq a_2 + \beta, \\ 0 & \text{otherwise,} \end{cases} \quad (2.9)$$

where the  $\gamma$ -level of  $A$  is defined by  $[A]^\gamma = [\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta]$  and its expected value is caculated by

$$E(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6}. \quad (2.10)$$

Then we have the following five cases.

*Case 1.*  $z < a_1 - \alpha$ . In this case we have  $E(A | z) = E(A)$ .

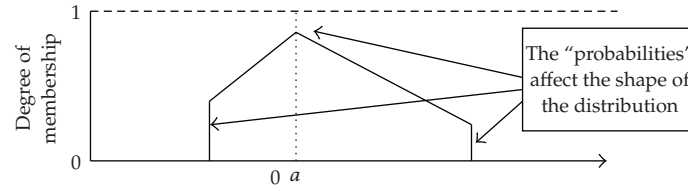
*Case 2.*  $a_1 - \alpha < z < a_1$ . Then introducing the notation

$$\gamma_z = \frac{z}{\alpha} - \frac{a_1 - \alpha}{\alpha}, \quad (2.11)$$

we find

$$[A]^\gamma = \begin{cases} (z, -\gamma\beta + a_2 + \beta) & \text{if } \gamma \leq \gamma_z, \\ (\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta) & \text{if } \gamma_z \leq \gamma \leq 1, \end{cases} \quad (2.12)$$

$$\begin{aligned} E(A | z) &= \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta) d\gamma + \int_{\gamma_z}^1 \gamma(\gamma\alpha + a_1 - \alpha - \gamma\beta + a_2 + \beta) d\gamma \\ &= \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6} + (z - a_1 + \alpha) \frac{\gamma_z^2}{2} - \alpha \frac{\gamma_z^3}{3}. \end{aligned} \quad (2.13)$$



**Figure 2:** Calculation of the fuzzy mean for the positive part of a fuzzy pay-off distribution of the form of special case.

Case 3.  $a_1 < z < a_2$ . In this case  $\gamma_z = 1$  and

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta], \quad (2.14)$$

and we get

$$\begin{aligned} E(A | z) &= \int_0^1 \gamma(z - \gamma\beta + a_2 + \beta) d\gamma \\ &= \frac{z + a_2}{2} + \frac{\beta}{6}. \end{aligned} \quad (2.15)$$

Case 4.  $a_2 < z < a_2 + \beta$ . In this case we have

$$\gamma_z = \frac{z}{-\beta} + c \frac{a_2 + \beta}{\beta}, \quad (2.16)$$

$$[A]^\gamma = [z, -\gamma\beta + a_2 + \beta], \quad (2.17)$$

if  $\gamma < \gamma_z$  and we find,

$$\begin{aligned} E(A | z) &= \int_0^{\gamma_z} \gamma(z - \gamma\beta + a_2 + \beta) d\gamma \\ &= (z + a_2 + \beta) \frac{\gamma_z^2}{2} - \beta \frac{\gamma_z^3}{3}. \end{aligned} \quad (2.18)$$

Case 5.  $a_2 + \beta < z$ . Then it is easy to see that  $E(A | z) = 0$ .

In the following special case, we expect that the managers will have already performed the construction of three cash-flow scenarios and have assigned estimated probabilities to each scenario (adding up to 100%). We want to use all this information and hence will assign the estimated “probabilities” to the scenarios resulting in a fuzzy number that has a graphical

presentation of the type presented in Figure 2 (not in scale):

$$\begin{aligned}
 A(u) &= \begin{cases} (\gamma_3 - \gamma_1) \frac{u}{\alpha} - (\gamma_3 - \gamma_1) \frac{a - \alpha}{\alpha} + \gamma_1 & \text{if } a - \alpha \leq u \leq a, \\ \gamma_3 & \text{if } u = a, \\ (\gamma_2 - \gamma_3) \frac{u}{\beta} - (\gamma_2 - \gamma_3) \frac{a}{\beta} + \gamma_3 & \text{if } a \leq u \leq a + \beta, \\ 0 & \text{otherwise,} \end{cases} \\
 E(A) &= \int_0^1 \gamma (a_1(\gamma) + a_2(\gamma)) d\gamma \\
 &= \int_0^1 \gamma a_1(\gamma) d\gamma + \int_0^1 \gamma a_2(\gamma) d\gamma, \\
 \int_0^1 \gamma a_1(\gamma) d\gamma &= \int_0^{\gamma_1} \gamma (a - \alpha) d\gamma + \int_{\gamma_1}^{\gamma_3} \gamma \left( \frac{\gamma - \gamma_1}{\gamma_3 - \gamma_1} \alpha + a - \alpha \right) d\gamma \quad (2.19) \\
 &= (a - \alpha) \frac{\gamma_1^2}{2} + \left( a - \alpha - \frac{\alpha \gamma_1}{\gamma_3 - \gamma_1} \right) \left( \frac{\gamma_3^2}{2} - \frac{\gamma_1^2}{2} \right) + \frac{\alpha}{\gamma_3 - \gamma_1} \left( \frac{\gamma_3^3}{3} - \frac{\gamma_1^3}{3} \right), \\
 \int_0^1 \gamma a_2(\gamma) d\gamma &= \int_0^{\gamma_2} \gamma (a + \beta) d\gamma + \int_{\gamma_2}^{\gamma_3} \gamma \left( \frac{\gamma - \gamma_3}{\gamma_2 - \gamma_3} \beta + a \right) d\gamma \\
 &= (a + \beta) \frac{\gamma_2^2}{2} + \left( a - \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) \left( \frac{\gamma_3^2}{2} - \frac{\gamma_2^2}{2} \right) + \frac{\beta}{\gamma_2 - \gamma_3} \left( \frac{\gamma_3^3}{3} - \frac{\gamma_2^3}{3} \right), \\
 E(A) &= \frac{\gamma_1^2}{2} \frac{\alpha \gamma_1}{\gamma_3 - \gamma_1} + \frac{\gamma_2^2}{2} \left( \beta + \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) + \frac{\gamma_3^2}{2} \left( 2a - \alpha - \frac{\alpha \gamma_1}{\gamma_3 - \gamma_1} - \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) \\
 &\quad - \frac{\gamma_1^3}{3} \frac{\alpha}{\gamma_3 - \gamma_1} - \frac{\gamma_2^3}{3} \frac{\beta}{\gamma_2 - \gamma_3} + \frac{\gamma_3^3}{3} \left( \frac{\alpha}{\gamma_3 - \gamma_1} + \frac{\beta}{\gamma_2 - \gamma_3} \right);
 \end{aligned}$$

$$(1) \ z < a - \alpha : E(A | z) = E(A),$$

$$(2) \ a - \alpha < z < a : \gamma_z = (\gamma_3 - \gamma_1) \frac{z}{\alpha} - (\gamma_3 - \gamma_1) \frac{a - \alpha}{\alpha} + \gamma_1,$$

$$\begin{aligned}
 E(A | z) &= \frac{\gamma_z^2}{2} \left( z - a + \alpha + \frac{\alpha \gamma_1}{\gamma_3 - \gamma_1} \right) + \frac{\gamma_z^2}{2} \left( \beta + \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) \\
 &\quad + \frac{\gamma_z^2}{2} \left( 2a - \alpha - \frac{\alpha \gamma_1}{\gamma_3 - \gamma_1} - \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) - \frac{\gamma_z^3}{3} \frac{\alpha}{\gamma_3 - \gamma_1} \\
 &\quad - \frac{\gamma_z^3}{3} \frac{\beta}{\gamma_2 - \gamma_3} + \frac{\gamma_z^3}{3} \left( \frac{\alpha}{\gamma_3 - \gamma_1} + \frac{\beta}{\gamma_2 - \gamma_3} \right), \quad (2.20)
 \end{aligned}$$

$$(3) \ a < z < a + \beta : \gamma_z = (\gamma_2 - \gamma_3) \frac{z}{\beta} - (\gamma_2 - \gamma_3) \frac{a}{\beta} + \gamma_3,$$

$$E(A | z) = \frac{\gamma_z^2}{2} \left( z + a - \frac{\beta}{\gamma_2 - \gamma_3} \right) + \frac{\gamma_2^2}{2} \left( \beta + \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} \right) + \frac{\gamma_z^3}{3} \frac{\beta \gamma_3}{\gamma_2 - \gamma_3} - \frac{\gamma_2^3}{3} \frac{\beta}{\gamma_2 - \gamma_3}, \quad (2.21)$$

$$(4) \ a + \beta < z : E(A | z) = 0.$$

In the same way as was discussed earlier in connection to the triangular NPV, to compute the real option value with the afore-mentioned formulas we must calculate the ratio between the positive area of the fuzzy number (NPV) and the total area of the same number according to the formula (2.1).

### 3. A Simple Case: Using the New Method in Analyzing a Business Case

The problem at hand is to evaluate the value of uncertain cash-flows from a business case. The input information available is in the form of three future cash-flow scenarios, good (optimistic), most likely, and bad (pessimistic). The same business case with the same numbers has been earlier presented in [7] and is presented here to allow superficial comparison with the Datar-Mathews method—we are using the same numbers with the fuzzy pay-off method.

The scenario values are given by managers as nonfuzzy numbers, they can, in general, have used any type of analysis tools, or models to reach these scenarios. For more accurate information on the generation of the numbers in this case, see [7] for reference. From the cost and benefit scenarios three scenarios for the NPV are combined (PV benefits - PV investment costs), where the cost cash-flows (CF) are discounted at the risk-free rate and the benefit CF discount rate is selected according to the risk (risk adjusted discount rate). The NPV is calculated for each of the three scenarios separately, see Figures 3 and 4. The resulting fuzzy NPV is the fuzzy pay-off distribution for the investment. To reach a similar probability distribution [7] use Monte Carlo simulation. They point out that a triangular distribution can also be used. The real option value for the investment can be calculated from the resulting fuzzy NPV, which is the pay-off distribution for the project, according to the formula presented in (2.1). We use the formula described in Section 2.1. to calculate the real option value for this business case. We reach the value ROV = 13.56. The work in [7] shows that the value with the same inputs is 8. The difference is caused by the difference in the distributions generated from the inputs.

It is usual that managers are asked to give cash-flow information in the form of scenarios (usually three) and they often have a preselected set of methods for building the scenarios. Usually the scenarios are constructed by trusting past experience and based on looking at, for example, the variables that most contribute to cash-flows and the future market outlook; similar approaches are also reported in [7].

With the fuzzy pay-off method, the scenario approach can be fully omitted and the future cash-flow forecasting can be done fully with fuzzy numbers. The end result will be a fuzzy NPV that is the pay-off distribution for the project. This is the same result that we get if we use scenarios, however, it does not require us to simplify the future to three alternative scenarios.

The detailed calculation used in the case includes present value calculation for the three scenarios of investment cost and revenue cash-flows and then integrates these to form

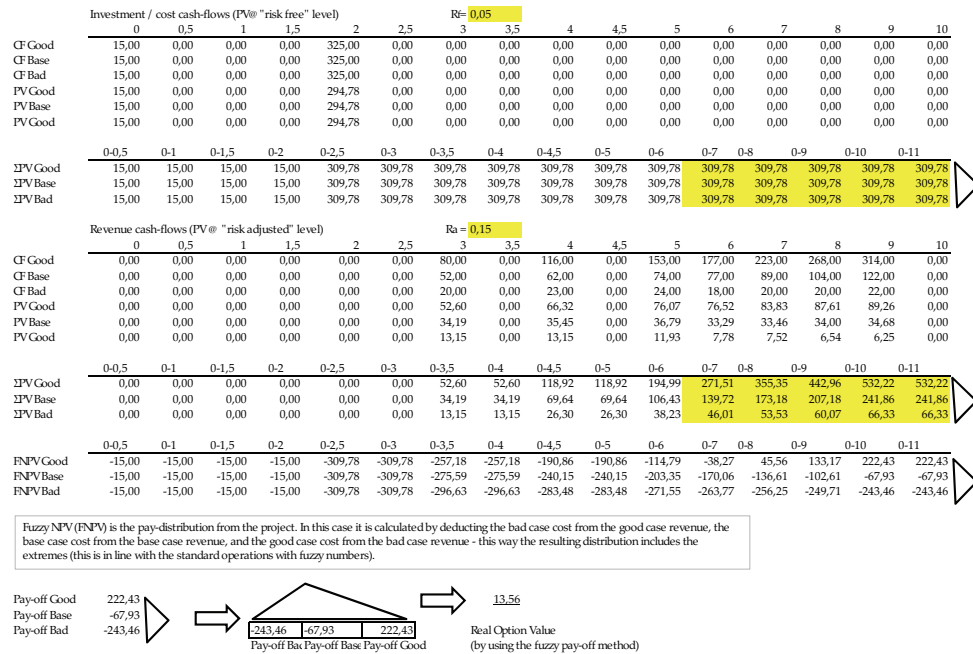


Figure 3: Detailed calculations used in the case.

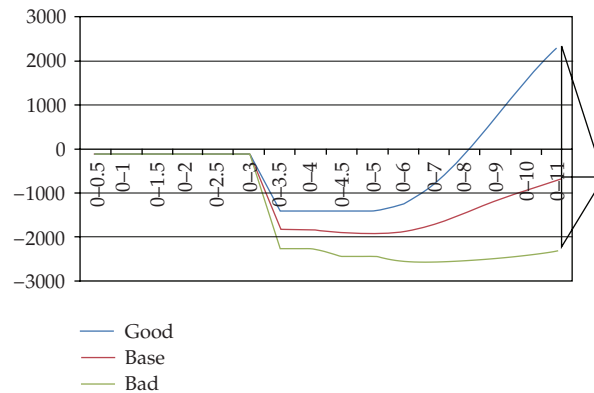
the fuzzy net present value (FNPV). The value of the R&D is directly included in the cost cash-flow table and the resulting ROV is what the work in [7] calls total project value. This is a minor issue, as the [7] project option value is the total project value + the R&D Cost.

#### 4. Discussion and Conclusions

There is a reason to expect that the simplicity of the presented method is an advantage over more complex methods. Using triangular and trapezoidal fuzzy numbers makes very easy implementations possible with the most commonly used spreadsheet software; this opens avenues for real option valuation to find its way to more practitioners. The method is flexible as it can be used when the fuzzy NPV is generated from scenarios or as fuzzy numbers from the beginning of the analysis. Fuzzy NPV is a distribution of the possible values that can take place for NPV; this means that it is by definition perceived as impossible at the time of the assessment that values outside of the number can happen. This is in line with the situation that real option value is zero when all the values of the fuzzy NPV are lower than zero. If we compare this to the presented case, we can see that in practice it is often that managers are not interested to use the full distribution of possible outcomes, but rather want to limit their assessment to the most possible alternatives (and leaving out the tails of the distribution). We think that the tails should be included in the real option analysis, because even remote possibilities should be taken into consideration.

The method brings forth an issue that has not gotten very much attention in academia, the dynamic nature of the assessment of investment profitability, that is, the assessment





**Figure 4:** Three NPV scenarios for the duration of the synergies that are used to generate (triangular) fuzzy NPV.

changes when information changes. As cash flows taking place in the future come closer, information changes, and uncertainty is reduced this should be reflected in the fuzzy NPV, the more there is uncertainty the wider the distribution should be, and when uncertainty is reduced, the width of the distribution should decrease. Only under full certainty should the distribution be represented by a single number, as the method uses fuzzy NPV there is a possibility to have the size of the distribution decrease with a lesser degree of uncertainty, this is an advantage vis-à-vis probability-based methods.

The common decision rules for ROV analysis are applicable with the ROV derived with the presented method. We suggest that the single number NPV needed for comparison purposes is derived from the (same) fuzzy NPV by calculating the fuzzy mean value. This means that in cases when all the values of the fuzzy NPV are greater than zero, the single number NPV equals ROV, which indicates immediate investment.

We feel that the presented new method opens possibilities for making simpler generic and modular real option valuation tools that will help construct real options analyses for systems of real options that are present in many types of investments.

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## Paper 3

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## A Fuzzy Pay-Off Method for Real Option Valuation: Credibilistic Approach

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**Abstract.** Real option analysis offers interesting insights on the value of assets and on the profitability of investments, which has made real options a growing field of academic research and practical application. Real option valuation is, however, often found to be difficult to understand and to implement due to the quite complex mathematics involved. Recent advances in modeling and analysis methods have made real option valuation easier to understand and to implement. This paper extends the results of our earlier paper on fuzzy pay-off method for real option valuation by using credibility measures. In: Gunalay Y (ed). Proceedings of the 3rd International Conference on Applied Operational Research – ICAOR (2011), pp xx–xx. Lecture Notes in Management Science Vol. 3. ISSN 2008-0050.

**Keywords:** Real Option Valuation, Fuzzy Real Options, Credibility Measure, Fuzzy Numbers.

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### 1 Introduction

Real option valuation (ROV) is treating investment opportunities and the different types of managerial flexibility as options and valuing them with option valuation models. Real options are useful both, as a mental model for strategic and operational decision-making, and as a valuation and numerical analysis tool. This paper concentrates on the use of real options in numerical analysis, and particularly on the derivation of the real option value for a given investment opportunity, or identified managerial flexibility.

Real options are commonly valued with the same methods that have been used to value financial options, i.e., with Black-Scholes option pricing formula (Black and Scholes, 1973), with the binomial option valuation method (Cox et al., 1979), with Monte-Carlo based methods (Boyle, 1977), and with a number of later methods based on these. Most of the methods are complex and demand a good understanding of the underlying mathematics, issues that make their use difficult in practice. In addition these models are based on the assumption that they can quite accurately mimic the underlying markets as a process, an assumption that may hold for some quite efficiently traded financial securities, but may not hold for real investments that do not have existing markets or have markets that can by no means be said to exhibit even weak market efficiency.

Recently, a novel approach to real option valuation was presented in Datar and Mathews, 2007, Mathews and Salmon, 2007, and in Datar and Mathews, 2004, where the real option value is calculated from a pay-off distribution, derived from a probability distribution of the NPV for a project that is generated with a (Monte-Carlo) simulation. The authors show that the results from the method converge to the results from the analytical Black-Scholes method. The method presented greatly simplifies the calculation of the real option value, making it more transparent and brings real option valuation as a method a big leap closer to practitioners. The most positive issue in this method is that it does not suffer from the problems associated with the assumptions connected to the market processes connected to the Black-Scholes and the binomial option valuation methods. The method utilizes cash-flow scenario based estimation of the future outcomes to derive the future pay-off distribution. This is highly compatible with the way cash-flow based profitability analysis is commonly done in companies.

All of the above mentioned models and methods use probability theory in their treatment of uncertainty, there are however, other ways than probability to treat uncertainty, or imprecision in future estimates, namely fuzzy logic and fuzzy sets. Fuzzy sets are sets that allow (have) gradation of belonging, such as "a future cash flow at year ten is about 10,000 euro". This means that fuzzy sets can be used to formalize inaccuracy that exists in human decision making and as a representation of vague, uncertain or imprecise knowledge, e.g., future cash-flow estimation, which human reasoning is especially adaptive to. "Fuzzy set-based methodologies blur the traditional line between qualitative and quantitative analysis, since the modeling may reflect more the type of information that is available rather than researchers' preferences" (Tarrazo, 1997) and indeed in economics "The use of fuzzy subsets theory leads to results that could not be obtained by classical methods" (Ponsard, 1988).

## 2 Credibility Measure

The concept of fuzzy set was introduced in 1965 by Zadeh (Zadeh, 1965), and later he proposed the concept of possibility measure (Zadeh 1978) to measure a fuzzy event. Although possibility measure has been widely used, it has no self-duality

property. This was the main motivation behind the concept of credibility measure which was first defined by Liu and Liu, 2002, where the authors used this subclass of fuzzy measures to define the expected value of a fuzzy random variable  $\xi$ . Later, credibility theory was founded by Liu (Liu, 2004), and later, Li and Liu, 2006, gave the following four axioms as a sufficient and necessary condition for a credibility measure ( $\Theta$  is a nonempty set and  $P(\Theta)$  is the power set of  $\Theta$ ):

- 1)  $Cr\{\Theta\}=1$
- 2)  $Cr$  is a Choquet capacity:  $Cr\{C\} \leq Cr\{D\}$  if  $C \subseteq D$
- 3)  $Cr$  is self-dual:  $Cr\{C\} + Cr\{\Theta \setminus C\} = 1$  for any  $C \in P(\Theta)$
- 4)  $Cr\{\bigcup_i C_i\} \wedge 0.5 = \sup_i Cr\{C_i\}$  for any  $C_i$  with  $Cr\{C_i\} \leq 0.5$

It is easy to see that the credibility of the empty set is 0, and  $0 \leq Cr\{C\} \leq 1$  for  $C \in P(\Theta)$ . The credibility measure is subadditive (see Liu, 2004),  $Cr\{C \cup D\} \leq Cr\{C\} + Cr\{D\}$  for any  $C, D \in P(\Theta)$ .

To establish the connection between a fuzzy variable and a credibility measure, both defined on the credibility space  $(\Theta, P(\Theta), Cr)$ , we can see a fuzzy variable,  $A$ , as a function from this space to the set of real numbers, and its membership function can be derived from the credibility measure by

$$\mu(x) = \min\{2Cr\{A = x\}, 1\},$$

for any  $x \in \mathbb{R}$ . We call  $\{A \in B\}$  a fuzzy event, where  $B$  is a set of real numbers. However, in practice a fuzzy variable is specified by its membership function. In this case we can calculate the credibility of fuzzy events by the credibility inversion theorem (Liu, 2004): Let  $A$  be a fuzzy variable with membership function  $\mu$ . Then for any set  $B$  of real numbers, we have

$$Cr\{A \in B\} = \frac{1}{2} \left( \sup_{x \in B} \mu(x) + 1 - \sup_{x \notin B} \mu(x) \right).$$

With this formula it is possible to interpret the credibility in terms of the possibility and necessity measure, since

$$Pos(B) = \sup_{x \in B} \mu(x), Nec(B) = 1 - \sup_{x \notin B} \mu(x).$$

Using this two measures, the theorem can be formulated as

$$Cr\{B\} = \frac{Pos(B) + Nec(B)}{2}. \quad (1)$$

We should note here that if one defines the credibility measure using the equation (1), then Li and Liu proved that this is equivalent to the definition in terms of the four axioms given above (Li and Liu, 2006).

In this paper we will only consider a special type of fuzzy variables, namely fuzzy numbers. There exist several definitions for fuzzy numbers, we will use the one introduced in Dubois and Prade, 1978: a fuzzy number  $A$  is a convex fuzzy set on the real line  $\mathbb{R}$  such that  $A$  is normal and it is piecewise continuous. Furthermore, we denote the family of all fuzzy numbers by  $F$ . A fuzzy set  $A$  of the real line  $\mathbb{R}$  is called triangular fuzzy number with peak (or center)  $a$ , left width  $\alpha > 0$  and right width  $\beta > 0$  if its membership function has the following form,

$$\mu_A(x) = \begin{cases} 1 - \frac{a-x}{\alpha} & \text{if } a - \alpha \leq x \leq a \\ 1 - \frac{x-a}{\beta} & \text{if } a \leq x \leq a + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation  $A = (a, \alpha, \beta)$ . From the definition of credibility measure, the credibility of the event  $A \leq x$  can be computed as,

$$Cr\{A \leq x\} = \begin{cases} 0 & \text{if } x \leq a - \alpha \\ \frac{1}{2} - \frac{a-x}{2\alpha} & \text{if } a - \alpha \leq x \leq a \\ \frac{1}{2} + \frac{x-a}{2\beta} & \text{if } a \leq x \leq a + \beta \\ 0 & \text{if } a + \beta \leq x \end{cases}$$

## 2.1 Expected value of normalized fuzzy variable

In Liu and Liu, 2002, the authors proposed a novel concept of expected value for normalized fuzzy variables motivated by the theory of Choquet integrals.

**Definition 1.** [Liu and Liu, 2002] The expected value of a normalized fuzzy variable,  $\xi$ , is defined by

$$E_C(\xi) = \int_0^\infty Cr\{\xi \geq r\} dr - \int_{-\infty}^0 Cr\{\xi \leq r\} dr$$

provided that at least one of the integrals is finite.

Let  $A = (a, \alpha, \beta)$  be a triangular fuzzy number. Then we find,

$$E_C(A) = a + \frac{\beta - \alpha}{4}$$

If  $A = (a_1, a_2, \alpha, \beta)$  is a trapezoidal fuzzy number defined by the membership function



$$\mu_A(x) = \begin{cases} 1 - \frac{a_1 - x}{\alpha} & \text{if } a_1 - \alpha \leq x \leq a_1 \\ 1 & \text{if } a_1 \leq x \leq a_2 \\ 1 - \frac{x - a_2}{\beta} & \text{if } a_2 \leq x \leq a_2 + \beta \\ 0 & \text{otherwise} \end{cases}$$

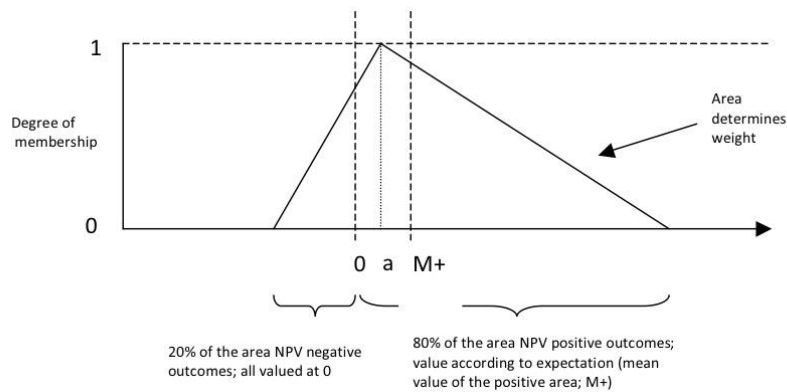
then its credibilistic expected value is,

$$E_C(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{4}$$

Credibility theory and specifically the credibilistic expected value have been applied to problems from different areas: portfolio optimization (Zhang et al., 2010), facility location problem in B2C e-commerce (Lau et al., 2010), transportation problems (Yang and Liu, 2007), logistics network design (Qin and Ji, 2010).

### 3 The fuzzy pay-off method

The fuzzy pay-off method was introduced in Collan et al., 2009 as a practical tool for the valuation of real options. Two recent papers (Mathews and Salmon, 2007, Datar and Mathews, 2007) present a practical probability theory-based Datar-Mathews method for the calculation of real option value and show that the method and results from the method are mathematically equivalent to the Black-Scholes formula (Black and Scholes, 1973). The DMM shows that the real option value can be understood as the probability-weighted average of the pay-off distribution. We use fuzzy numbers in representing the expected future distribution of possible project costs and revenues, and hence also the profitability (NPV) outcomes. The fuzzy NPV, a fuzzy number, is the pay-off distribution from the project (see Fig. 1).



**Fig. 1** Fuzzy NPV

The method presented in Datar and Mathews, 2007, implies that the weighted average of the positive outcomes of the pay-off distribution is the real option value, in the case with fuzzy numbers the weighted average is the fuzzy mean value of the positive NPV outcomes. This means that calculating the ROV from a fuzzy NPV (distribution) is straightforward, it is the fuzzy mean of the possibility distribution with values below zero counted as zero, i.e., the area weighted average of the fuzzy mean of the positive values of the distribution and zero (for negative values).

**Definition 2.** We calculate the real option value from the fuzzy NPV as follows

$$ROV = \frac{\int_{-\infty}^{\infty} A(x) dx}{\int_{-\infty}^{\infty} A(x) dx} \times E(A_+)$$

where  $A$  stands for the fuzzy NPV,  $E(A_+)$  denotes the fuzzy mean value of the positive side of the NPV, the integral in the denominator computes the area below the whole fuzzy number  $A$ , and the integral in the numerator computes the area below the positive part of  $A$ .

For fuzzy numbers, there are many ways to define an expected value operator, for example Dubois and Prade, 1987, Heilpern, 1992, Yager, 1981. In Collan et al., 2009, we used the possibilistic mean value to calculate the expected value of the positive side of the NPV:

**Definition 3.** The possibilistic (or fuzzy) mean value of the fuzzy number  $A$  with  $(A)^\gamma = (a_1(\gamma), a_2(\gamma))$  is defined in Carlsson and Fullér, 2001 by

$$E_P(A) = \int_0^1 2\gamma \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) d\gamma.$$

To use the credibility measure and the credibilistic expected value in this real option environment seems to be a natural choice. To compare the results with the possibilistic mean value, we will use the same examples from Collan et al., 2009. In case of credibilistic expected value, the calculation of the mean of the positive part means that we need to use

$$E_C(A_+) = \int_0^{\infty} Cr\{A \geq r\} dr$$

When we calculate the positive area and the mean of the positive area of a triangular fuzzy pay-off, we have five possible cases:

*Case 1:*  $0 < a - \alpha$ . In this case we have

$$E_C(A_+) = E_C(A) = a + \frac{\beta - \alpha}{4}$$

We note here that the possibilistic mean value of a triangular fuzzy number is  $a + \frac{\beta - \alpha}{6}$ . Comparing this value to the result above, we can observe that

$$|E_P(A) - a| \leq |E_C(A) - a|.$$

Also important to note, that  $E_P(A) \leq E_C(A)$  if and only if the left width,  $\alpha$  is smaller than the right width,  $\beta$ .

Case 2:  $a - \alpha \leq 0 \leq a$ . Then the credibilistic expected value has the following form:

$$\begin{aligned} E_C(A_+) &= \int_0^\infty Cr\{A \geq r\} dr = \int_0^a \left( \frac{1}{2} + \frac{a-r}{2\alpha} \right) dr + \int_a^{a+\beta} \left( \frac{1}{2} - \frac{r-a}{2\beta} \right) dr \\ &= \frac{a}{2} + \frac{a^2}{4\alpha} + \frac{\beta}{4} \end{aligned}$$

Case 3:  $a \leq 0 \leq a + \beta$ . In this case

$$E_C(A_+) = \int_0^\infty Cr\{A \geq r\} dr = \int_0^{a+\beta} \left( \frac{1}{2} - \frac{r-a}{2\beta} \right) dr = \frac{a}{2} + \frac{a^2}{4\beta} + \frac{\beta}{4}$$

Case 4:  $a + \beta \leq 0$ . Then it is easy to see that  $E_C(A_+) = 0$ .

For computing the real option value from an NPV (pay-off) distribution of a trapezoidal form we must consider a trapezoidal fuzzy pay-off distribution  $A$  defined by

$$A(x) = \begin{cases} \frac{x - a_1 - \alpha}{\alpha} & \text{if } a_1 - \alpha \leq x \leq a_1 \\ 1 & \text{if } a_1 \leq x \leq a_2 \\ -\frac{x}{\beta} + \frac{\beta + a_2}{\beta} & \text{if } a_2 \leq x \leq a_2 + \beta \\ 0 & \text{otherwise} \end{cases}$$

where the  $\gamma$ -level of  $A$  is defined by

$$(A)^\gamma = (\gamma\alpha + a_1 - \alpha, -\gamma\beta + a_2 + \beta)$$

In trapezoidal case the credibility has the following form:

$$Cr\{A \leq x\} = \begin{cases} 0 & \text{if } x \leq a_1 - \alpha \\ \frac{1}{2} - \frac{a_1 - x}{2\alpha} & \text{if } a_1 - \alpha \leq x \leq a_1 \\ \frac{1}{2} & \text{if } a_1 \leq x \leq a_2 \\ \frac{1}{2} + \frac{x - a_2}{2\beta} & \text{if } a_2 \leq x \leq a_2 + \beta \\ 0 & \text{if } a_2 + \beta \leq x \end{cases}$$

Then to calculate the credibilistic expected value for the positive part, we need to consider the following five cases:

*Case 1:*  $0 < a_1 - \alpha$ . In this case we have

$$E_C(A_+) = E_C(A) = \frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{4}$$

We note here that the possibilistic mean value of a triangular fuzzy number is  $\frac{a_1 + a_2}{2} + \frac{\beta - \alpha}{6}$ . Comparing this value to the result above, we can observe that

$$\left| E_P(A) - \frac{a_1 + a_2}{2} \right| \leq \left| E_C(A) - \frac{a_1 + a_2}{2} \right|.$$

Also important to note, that  $E_P(A) \leq E_C(A)$  if and only if the left width,  $\alpha$  is smaller than the right width,  $\beta$ .

*Case 2:*  $a_1 - \alpha \leq 0 \leq a_1$ . Then the credibilistic expected value has the following form:

$$\begin{aligned} E_C(A_+) &= \int_0^{a_1} \left( \frac{1}{2} + \frac{a_1 - r}{2\alpha} \right) dr + \int_{a_1}^{a_2} \frac{1}{2} dr + \int_{a_2}^{a_2 + \beta} \left( \frac{1}{2} - \frac{r - a_2}{2\beta} \right) dr \\ &= \frac{a_2}{2} + \frac{a_1^2}{4\alpha} + \frac{\beta}{4} \end{aligned}$$

*Case 3:*  $a_1 \leq 0 \leq a_2$ . In this case

$$E_C(A_+) = \int_0^{a_2} \frac{1}{2} dr + \int_{a_2}^{a_2 + \beta} \left( \frac{1}{2} - \frac{r - a_2}{2\beta} \right) dr = \frac{a_2}{2} + \frac{\beta}{4}$$

*Case 4:*  $a_2 \leq 0 \leq a_2 + \beta$ . In this case we have

$$E_c(A_+) = \int_0^{a_2+\beta} \left( \frac{1}{2} - \frac{r-a_2}{2\beta} \right) dr = \frac{a_2}{2} + \frac{a_2^2}{4\beta} + \frac{\beta}{4}$$

Case 5:  $a_2 + \beta \leq 0$ . Then it is easy to see that  $E_c(A_+) = 0$ .

## 4 Conclusions

There is reason to expect that the simplicity of the fuzzy pay-off method (Collan et al., 2009) is an advantage over more complex methods. Using triangular and trapezoidal fuzzy numbers make very easy implementations possible with the most commonly used spreadsheet software; this opens avenues for real option valuation to find its way to more practitioners. The method is flexible as it can be used when the fuzzy NPV is generated from scenarios or as fuzzy numbers from the beginning of the analysis. Fuzzy NPV is a distribution of the possible values that can take place for NPV; this means that it is by definition perceived as impossible at the time of the assessment that values outside of the number can happen - this is in line with the situation that real option value is zero when all the values of the fuzzy NPV are lower than zero. The calculation of the real option value in this method is based on the mean value of a fuzzy number. In this paper we compared the originally used possibilistic mean value with the the credibilistic expectation and showed the differences for two classes of fuzzy numbers: triangular and trapezoidal.

Future work will be focused on two aspects. Firstly, we can use other different fuzzy mean value concepts and compare the results. Secondly, the application of this method with quasi fuzzy numbers (fuzzy numbers with infinite support) will be examined.

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## Paper 4

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# An improved index of interactivity for fuzzy numbers

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## Abstract

In this paper we will introduce a new index of interactivity between marginal possibility distributions  $A$  and  $B$  of a joint possibility distribution  $C$ . The starting point of our approach is to equip each  $\gamma$ -level set of  $C$  with a uniform probability distribution, then the probabilistic correlation coefficient between its marginal probability distributions is interpreted as an index of interactivity between the  $\gamma$ -level sets of  $A$  and  $B$ . Then we define the index of interactivity between  $A$  and  $B$  as the weighted average of these indexes over the set of all membership grades. This new index of interactivity is meaningful for the whole family of joint possibility distributions. © 2010 Elsevier B.V. All rights reserved.

**Keywords:** Possibility distribution; Interactive fuzzy numbers; Possibilistic correlation; Uniform probability distribution

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## 1. Introduction

In probability theory the notion of expected value of functions of random variables plays a fundamental role in defining the basic characteristic measures of probability distributions. For instance, the measure of covariance, variance and correlation of random variables can all be computed as probabilistic means of their appropriately chosen real-valued functions. For expected value, variance, covariance and correlation of *fuzzy random variables* the reader can consult, e.g. Kwakernaak [13,14], Puri and Ralescu [18], Körner [15], Watanabe and Imaizumi [22], Feng et al. [8], Näther [17] and Shapiro [19].

In possibility theory we can use the principle of *average value* of appropriately chosen real-valued functions to define mean value, variance, covariance and correlation of possibility distributions. A function  $f: [0, 1] \rightarrow \mathbb{R}$  is said to be a weighting function if  $f$  is non-negative, monotone increasing and satisfies the following normalization condition  $\int_0^1 f(\gamma) d\gamma = 1$ . Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions. We can define the mean value (variance) of a possibility distribution as the  $f$ -weighted average of the probabilistic mean values (variances) of the respective uniform distributions defined on the  $\gamma$ -level sets of that possibility distribution. A measure of possibilistic covariance between marginal possibility distributions of a joint possibility distribution can be defined as the  $f$ -weighted average of probabilistic covariances between marginal

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probability distributions whose joint probability distribution is defined to be uniform on the  $\gamma$ -level sets of their joint possibility distribution [10]. This is an absolute measure of interactivity. A measure of possibilistic correlation between marginal possibility distributions of a joint possibility distribution can be defined as their possibilistic covariance divided by the square root of the product of their possibilistic variances [2]. This is a relative measure of interactivity. We should note here that the choice of uniform probability distribution on the level sets of possibility distributions is not without reason. We suppose that each point of a given level set is equally possible and then we apply Laplace's principle of Insufficient Reason: if elementary events are equally possible, they should be equally probable (for more details and generalization of principle of Insufficient Reason see [7, p. 59]). The idea of equipping the alpha-cuts with a uniform probability is not new and refers to early ideas of simulation of fuzzy sets by Yager [23], and possibility/probability transforms by Dubois et al. [5] as well as the pignistic transform of Smets [20].

The main drawback of the measure of possibilistic correlation introduced in [2] that it does not necessarily take its values from  $[-1, 1]$  if some level-sets of the joint possibility distribution are not convex. A new normalization technique is needed.

In this paper we will introduce a new index of interactivity between marginal distributions of a joint possibility distribution, which is defined for the whole family of joint possibility distributions. Namely, we will equip each level set of a joint possibility distribution with a uniform probability distribution, then compute the probabilistic correlation coefficient between its marginal probability distributions, and then the new index of interactivity is computed as the weighted average of these coefficients over the set of all membership grades. These weights (or importances) can be given by weighting functions.

A *fuzzy number*  $A$  is a fuzzy set in  $\mathbb{R}$  with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by  $\mathcal{F}$ . Fuzzy numbers can be considered as possibility distributions. A  $\gamma$ -level set of a fuzzy set  $A$  in  $\mathbb{R}^m$  is defined by  $[A]^\gamma = \{x \in \mathbb{R}^m : A(x) \geq \gamma\}$  if  $\gamma > 0$  and  $[A]^0 = \text{cl}\{x \in \mathbb{R}^m : A(x) > 0\}$ . A joint possibility distribution of fuzzy numbers is defined as a normal fuzzy set  $C$  in  $\mathbb{R}^2$ . Furthermore,  $A$  and  $B$  are called the marginal possibility distributions of  $C$  if it satisfies the relationships

$$\max\{y \in \mathbb{R} \mid C(x, y)\} = A(x) \quad \text{and} \quad \max\{x \in \mathbb{R} \mid C(x, y)\} = B(y),$$

for all  $x, y \in \mathbb{R}$ . In the following we will suppose that  $C$  is given in such a way that a uniform distribution can be defined on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ . Marginal possibility distributions are always uniquely defined from their joint possibility distribution by the *principle of falling shadows*.

Let  $C$  be a joint possibility distribution with marginal possibility distributions  $A, B \in \mathcal{F}$ , and let  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$  and  $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ ,  $\gamma \in [0, 1]$ . Then  $A$  and  $B$  are said to be non-interactive if their joint possibility distribution is  $A \times B$ ,

$$C(x, y) = \min\{A(x), B(y)\},$$

for all  $x, y \in \mathbb{R}$ , which can be written in the form,  $[C]^\gamma = [A]^\gamma \times [B]^\gamma$ , that is,  $[C]^\gamma$  is rectangular subset of  $\mathbb{R}^2$ , for any  $\gamma \in [0, 1]$ . If  $A$  and  $B$  are non-interactive then for any  $x \in [A]^\gamma$  and any  $y \in [B]^\gamma$  we have that the ordered pair  $(x, y)$  will be in  $[C]^\gamma$  for any  $\gamma \in [0, 1]$ . In other words, if one takes a point,  $x$ , from the  $\gamma$ -level set of  $A$  and then takes an arbitrarily chosen point,  $y$ , from the  $\gamma$ -level set of  $B$  then the pair  $(x, y)$  will belong to the  $\gamma$ -level set of  $C$ .

Another extreme situation is when  $[C]^\gamma$  is a line segment in  $\mathbb{R}^2$ . For example, let  $[0, 1] \times [0, 1]$  be the universe of discourse for  $C$  and let, the *diagonal beam*,

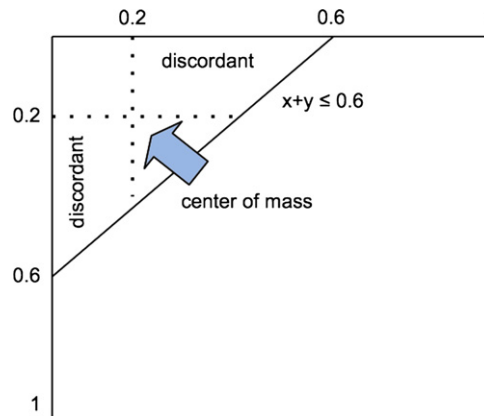
$$C(x, y) = x \chi_{\{x=y\}}(x, y),$$

for any  $x, y \in [0, 1]$ , be the joint possibility distribution of marginal possibility distributions  $A(x) = x$  and  $B(y) = y$ . Then  $[C]^\gamma$  is a line segment  $[(\gamma, \gamma), (1, 1)]$  in  $\mathbb{R}^2$  for any  $\gamma \in [0, 1]$ . Furthermore, if one takes a point,  $x$ , from the  $\gamma$ -level set of  $A$  then one can take only  $y = x$  from the  $\gamma$ -level set of  $B$  for the pair  $(x, y)$  to belong to  $[C]^\gamma$ . This point-to-point interactivity relation is the strongest one that we can envisage between  $\gamma$ -level sets of marginal possibility distributions.

What can one say about the strength of interactivity between marginal distributions,  $A(x) = 1 - x$  and  $B(y) = 1 - y$ , when their joint distribution,  $F$ , is defined, for example, by the Lukasiewicz t-norm? In this case

$$F(x, y) = \max\{A(x) + B(y) - 1, 0\} = \max\{1 - x + 1 - y - 1, 0\} = \max\{1 - x - y, 0\},$$

and  $[F]^\gamma = \{(x, y) \mid x + y \leq 1 - \gamma\}$  is of symmetric triangular form for any  $0 \leq \gamma < 1$ . If we take, for example,  $\gamma = 0.4$  then the pair  $(0.3, 0.2)$  belongs to  $[F]^{0.4}$  since  $0.3 + 0.2 \leq 1 - 0.4$ , but the pair  $(0.4, 0.4)$  does not

Fig. 1. Illustration of  $[F]^{0.4}$ .

(see Fig. 1). In our approach we will define a uniform probability distribution on  $[F]^{0.4}$  with marginal probability distributions denoted by  $X_{0.4}$  and  $Y_{0.4}$ . The expected value of this uniform probability distribution,  $(0.2, 0.2)$ , will be nothing else but the center of mass (or gravity) of the set  $[F]^{0.4}$  of homogeneous density (for calculations see Section 4). Then the probabilistic correlation coefficient, denoted by  $\rho(X_{0.4}, Y_{0.4})$ , will be negative since the ‘strength’ of pairs  $(x, y) \in [F]^{0.4}$  that are discordant (i.e.  $(x - 0.2)(y - 0.2) < 0$ ) is bigger than the ‘strength’ of those ones that are concordant (i.e.  $(x - 0.2)(y - 0.2) > 0$ ). Then we define the index of interactivity as the weighted average of these correlation coefficients over the set of all membership grades.

Let  $A \in \mathcal{F}$  be fuzzy number with a  $\gamma$ -level set denoted by  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ,  $\gamma \in [0, 1]$  and let  $U_\gamma$  denote a uniform probability distribution on  $[A]^\gamma$ ,  $\gamma \in [0, 1]$ . Recall that the mean value of  $U_\gamma$  is  $M(U_\gamma) = (a_1(\gamma) + a_2(\gamma))/2$  and its variance is computed by  $\text{var}(U_\gamma) = (a_2(\gamma) - a_1(\gamma))^2/12$ .

## 2. Possibilistic mean value, variance, covariance and correlation

The  $f$ -weighted possibilistic mean value of a possibility distribution  $A \in \mathcal{F}$  is the  $f$ -weighted average of probabilistic mean values of the respective uniform distributions on the level sets of  $A$ . That is, the  $f$ -weighted *possibilistic mean value* of  $A \in \mathcal{F}$ , with  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ,  $\gamma \in [0, 1]$ , is defined by [9]

$$E_f(A) = \int_0^1 M(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma, \quad (1)$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$  for all  $\gamma \in [0, 1]$ . This definition is based on Goetschel–Voxman ordering of fuzzy numbers [11], and it can be considered as a particular case of the average index proposed by Campos and González in [1].

If  $f(\gamma) \equiv 1$  the  $f$ -weighted possibilistic mean value coincides with the (i) generative expectation of fuzzy numbers introduced by Chanas and Nowakowski in [3, p. 47]; (ii) middle-point-of-the-mean-interval defuzzification method proposed by Yager in [23, p. 161].

**Note 1.** There exist several other ways to define mean values of fuzzy numbers, e.g. Dubois and Prade [4] defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation remains additive in the sense of addition of fuzzy numbers. Using evaluation measures, Yoshida et al. [24] introduced a possibility mean, a necessity mean and a credibility mean of fuzzy numbers that are different from (1). Surveying the results in quantitative possibility theory, Dubois [7] showed that some notions (e.g. cumulative distributions, mean values) in statistics can naturally be interpreted in the language of possibility theory.

The  $f$ -weighted possibilistic covariance between marginal possibility distributions of a joint possibility distribution is defined as the  $f$ -weighted average of probabilistic covariances between marginal probability distributions whose

joint probability distribution is uniform on each level-set of the joint possibility distribution. That is, the  $f$ -weighted *possibilistic covariance* between  $A, B \in \mathcal{F}$ , (with respect to their joint distribution  $C$ ), can be written as [10]

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\text{cov}(X_\gamma, Y_\gamma)$  denotes their probabilistic covariance. It should be noted that the possibilistic covariance is an absolute measure of interactivity between marginal possibility distributions.

The measure of  $f$ -weighted possibilistic variance of  $A$  is the  $f$ -weighted average of the probabilistic variances of the respective uniform distributions on the level sets of  $A$ . That is, the  $f$ -weighted *possibilistic variance* of  $A$  is defined as [10]

$$\text{Var}_f(A) = \int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma = \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

There exist other approaches to define variance of fuzzy numbers, e.g. Dubois et al. [6] defined the potential variance of a symmetric fuzzy interval by viewing it as a family of its  $\alpha$ -cut.

A measure of possibilistic correlation between marginal possibility distributions  $A$  and  $B$  of a joint possibility distribution  $C$  has been defined in [2] as their possibilistic covariance divided by the square root of the product of their possibilistic variances. That is, the  $f$ -weighted measure of *possibilistic correlation* of  $A, B \in \mathcal{F}$  (with respect to their joint distribution  $C$ ) is,

$$\begin{aligned} \rho_f^{\text{old}}(A, B) &= \frac{\text{Cov}_f(A, B)}{\sqrt{\text{Var}_f(A)} \sqrt{\text{Var}_f(B)}} \\ &= \frac{\int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma}{\left( \int_0^1 \text{var}(U_\gamma) f(\gamma) d\gamma \right)^{1/2} \left( \int_0^1 \text{var}(V_\gamma) f(\gamma) d\gamma \right)^{1/2}}, \end{aligned} \quad (2)$$

where  $U_\gamma$  is a uniform probability distribution on  $[A]^\gamma$ , and  $V_\gamma$  is a uniform probability distribution on  $[B]^\gamma$ . Thus, the possibilistic correlation represents an average degree to which  $X_\gamma$  and  $Y_\gamma$  are linearly associated as compared to the dispersions of  $U_\gamma$  and  $V_\gamma$ . We have the following result [2]. If  $[C]^\gamma$  is convex for all  $\gamma \in [0, 1]$  then  $-1 \leq \rho_f^{\text{old}}(A, B) \leq 1$  for any  $f$ .

The presence of weighting function is not crucial in our theory: we can simply remove it from consideration by choosing  $f(\gamma) \equiv 1$ .

**Note 2.** There exist several other ways to define correlation coefficient for fuzzy numbers, e.g. Liu and Kao [16] used fuzzy measures to define a fuzzy correlation coefficient of fuzzy numbers and they formulated a pair of nonlinear programs to find the  $\alpha$ -cut of this fuzzy correlation coefficient, then, in a special case, Hong [12] showed an exact calculation formula for this fuzzy correlation coefficient. Vaidyanathan [21] introduced a new measure for the correlation coefficient between triangular fuzzy variables called credibilistic correlation coefficient.

### 3. An improved index of interactivity for fuzzy numbers

The main drawback of the definition of the former index of interactivity (2) is that it does not necessarily take its values from  $[-1, 1]$  if some level-sets of the joint possibility distribution are not convex. For example, consider a joint possibility distribution defined by

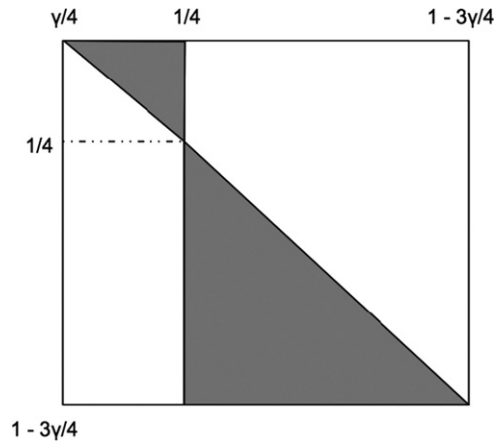
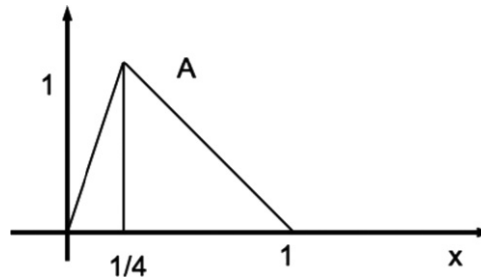
$$C(x, y) = 4x \cdot \chi_T(x, y) + 4/3(1 - x) \cdot \chi_S(x, y), \quad (3)$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1/4, 0 \leq y \leq 1/4, x \leq y\},$$

and

$$S = \{(x, y) \in \mathbb{R}^2 \mid 1/4 \leq x \leq 1, 1/4 \leq y \leq 1, y \leq x\}.$$

Fig. 2. Not convex  $\gamma$ -level set.Fig. 3. Marginal distribution  $A$ .

Furthermore, we have

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid \gamma/4 \leq x \leq 1/4, x \leq y \leq 1/4\} \\ \cup \{(x, y) \in \mathbb{R}^2 \mid 1/4 \leq x \leq 1 - 3\gamma/4, 1/4 \leq y \leq x\}.$$

We can see that  $[C]^\gamma$  is not a convex set for any  $\gamma \in [0, 1]$  (see Fig. 2).

Then the marginal possibility distributions of (3) are computed by (see Fig. 3)

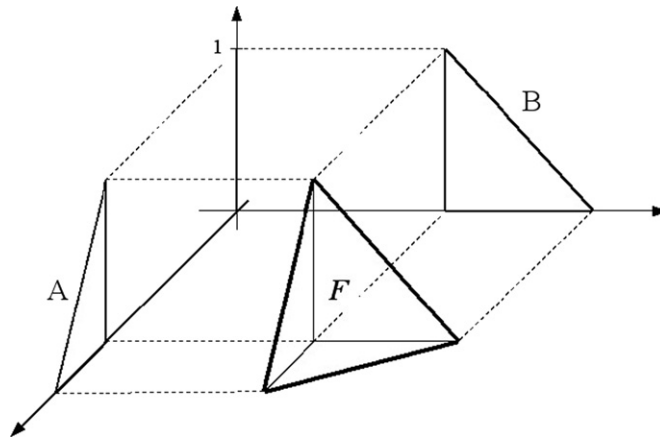
$$A(x) = B(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/4, \\ \frac{4}{3}(1-x) & \text{if } 1/4 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

After some computations we get  $\rho_f^{\text{old}}(A, B) \approx 1.562$  for the weighting function  $f(\gamma) = 2\gamma$ . We get here a value bigger than one since the variance of the first marginal distributions,  $X_\gamma$ , exceeds the variance of the uniform distribution on the same support.

Let us now introduce a new index of interactivity between marginal distributions  $A$  and  $B$  of a joint possibility distribution  $C$  as the  $f$ -weighted average of the probabilistic correlation coefficients between the marginal probability distributions of a uniform probability distribution on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ . That is,

**Definition 1.** The  $f$ -weighted index of interactivity of  $A, B \in \mathcal{F}$  (with respect to their joint distribution  $C$ ) is defined by

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma, \quad (4)$$

Fig. 4. Illustration of joint possibility distribution  $F$ .

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}}$$

and, where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ .

In other words, the ( $f$ -weighted) index of interactivity is nothing else, but the  $f$ -weighted average of the probabilistic correlation coefficients  $\rho(X_\gamma, Y_\gamma)$  for all  $\gamma \in [0, 1]$ . It is clear that for any joint possibility distribution this new correlation coefficient always takes its value from interval  $[-1, 1]$ , since  $\rho(X_\gamma, Y_\gamma) \in [-1, 1]$  for any  $\gamma \in [0, 1]$  and  $\int_0^1 f(\gamma) d\gamma = 1$ . As for the joint possibility distribution defined by (3) we get  $\rho_f(A, B) \approx 0.786$  for any  $f$ . Since  $\rho_f(A, B)$  measures an average index of interactivity between the level sets of  $A$  and  $B$ , we sometimes will call this measure as the  $f$ -weighted possibilistic correlation coefficient.

#### 4. An example

Consider the case, when  $A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, 1-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by  $F(x, y) = (1-x-y) \cdot \chi_T(x, y)$ , where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}.$$

A  $\gamma$ -level set of  $F$  is computed by

$$[F]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1 - \gamma\}.$$

This situation is depicted in Fig. 4, where we have shifted the fuzzy sets to get a better view of the situation. The density function of a uniform distribution on  $[F]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[F]^\gamma} dx dy} & \text{if } (x, y) \in [F]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{(1-\gamma)^2} & \text{if } (x, y) \in [F]^\gamma \\ 0 & \text{otherwise.} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(1-\gamma-x)}{(1-\gamma)^2} & \text{if } 0 \leq x \leq 1-\gamma, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{2(1-\gamma-y)}{(1-\gamma)^2} & \text{if } 0 \leq y \leq 1-\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[F]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} x(1-\gamma-x) dx = \frac{1-\gamma}{3}$$

and,

$$M(Y_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} y(1-\gamma-y) dy = \frac{1-\gamma}{3}.$$

We calculate the variations of  $X_\gamma$  and  $Y_\gamma$  with the formula  $\text{var}(X) = M(X^2) - M(X)^2$ :

$$M(X_\gamma^2) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} x^2(1-\gamma-x) dx = \frac{(1-\gamma)^2}{6}$$

and,

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^2}{6} - \frac{(1-\gamma)^2}{9} = \frac{(1-\gamma)^2}{18}.$$

And similarly we obtain

$$\text{var}(Y_\gamma) = \frac{(1-\gamma)^2}{18}.$$

Using that

$$M(X_\gamma Y_\gamma) = \frac{2}{(1-\gamma)^2} \int_0^{1-\gamma} \int_0^{1-\gamma-x} xy dy dx = \frac{(1-\gamma)^2}{12},$$

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(1-\gamma)^2}{36},$$

we can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$ :

$$\rho_f(A, B) = \int_0^1 -\frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

We note here that using the former definition (2) we would obtain  $\rho_f^{\text{old}}(A, B) = -1/3$  for the correlation coefficient (see [2] for details).

## 5. Some important examples

In this section we will show three important examples for the possibilistic correlation coefficient.

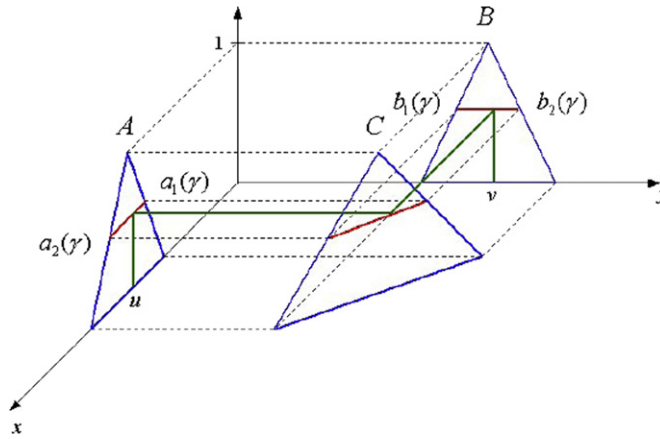


Fig. 5. Perfect negative correlation.

### 5.1. Non-interactive fuzzy numbers

If  $A$  and  $B$  are non-interactive then their joint possibility distribution is defined by  $C = A \times B$ . Since all  $[C]^\gamma$  are rectangular and the probability distribution on  $[C]^\gamma$  is defined to be uniform we get  $\text{cov}(X_\gamma, Y_\gamma) = 0$ , for all  $\gamma \in [0, 1]$ . So  $\text{Cov}_f(A, B) = 0$  and  $\rho_f(A, B) = 0$  for any weighting function  $f$ .

### 5.2. Perfect correlation

Fuzzy numbers  $A$  and  $B$  are said to be in perfect correlation, if there exist  $q, r \in \mathbb{R}$ ,  $q \neq 0$  such that their joint possibility distribution is defined by [2]

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2) = B(x_2) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2), \quad (5)$$

where  $\chi_{\{qx_1 + r = x_2\}}$  stands for the characteristic function of the line

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid qx_1 + r = x_2\}.$$

In this case we have

$$[C]^\gamma = \{(x, qx + r) \in \mathbb{R}^2 \mid x = (1 - t)a_1(\gamma) + ta_2(\gamma), t \in [0, 1]\},$$

where  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ; and  $[B]^\gamma = q[A]^\gamma + r$ , for any  $\gamma \in [0, 1]$ , and, finally,

$$B(x) = A\left(\frac{x - r}{q}\right),$$

for all  $x \in \mathbb{R}$ . Furthermore,  $A$  and  $B$  are in a perfect positive (see Fig. 6) (negative [see Fig. 5]) correlation if  $q$  is positive (negative) in (5).

If  $A$  and  $B$  have a perfect positive (negative) correlation then from  $\rho(X_\gamma, Y_\gamma) = 1$  ( $\rho(X_\gamma, Y_\gamma) = -1$ ) (see [2] for details), for all  $\gamma \in [0, 1]$ , we get  $\rho_f(A, B) = 1$  ( $\rho_f(A, B) = -1$ ) for any weighting function  $f$ .

### 5.3. Mere shadows

Suppose that the joint possibility distribution of  $A$  and  $B$  is defined by

$$C(x, y) = \begin{cases} A(x) & \text{if } y = 0, \\ B(y) & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$



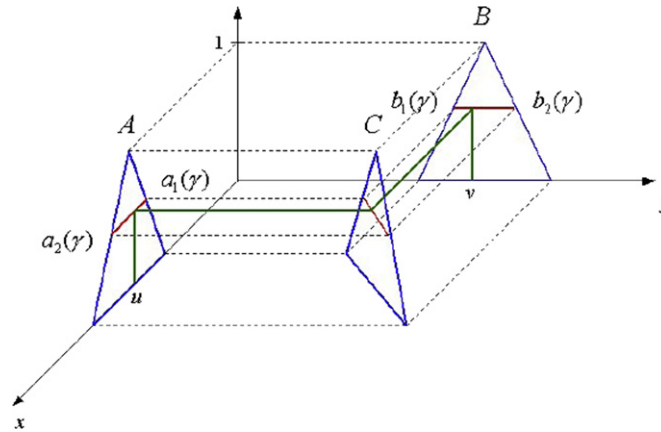


Fig. 6. Perfect positive correlation.

Suppose further that,

$$A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ . Then a  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 - \gamma\} \cup \{(0, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 - \gamma\}.$$

Since all  $\gamma$ -level sets of  $C$  are degenerated, i.e. their integrals vanish, we calculate everything as a limit of integrals. We calculate all the quantities with the  $\gamma$ -level sets:

$$[C]_\delta^\gamma = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 - \gamma, 0 \leq y \leq \delta\} \\ \cup \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 - \gamma, 0 \leq x \leq \delta\}.$$

First we calculate the expected value and variance of  $X_\gamma$  and  $Y_\gamma$ :

$$M(X_\gamma) = \lim_{\delta \rightarrow 0} \frac{1}{\int_{[C]_\delta^\gamma} dx dy} \int_{[C]_\delta^\gamma} x dx = \frac{1 - \gamma}{4},$$

$$M(X_\gamma^2) = \lim_{\delta \rightarrow 0} \frac{1}{\int_{[C]_\delta^\gamma} dx dy} \int_{[C]_\delta^\gamma} x^2 dx = \frac{(1 - \gamma)^2}{6},$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1 - \gamma)^2}{6} - \frac{(1 - \gamma)^2}{16} = \frac{5(1 - \gamma)^2}{48}.$$

Because of the symmetry, the results are the same for  $Y_\gamma$ . We need to calculate their covariance,

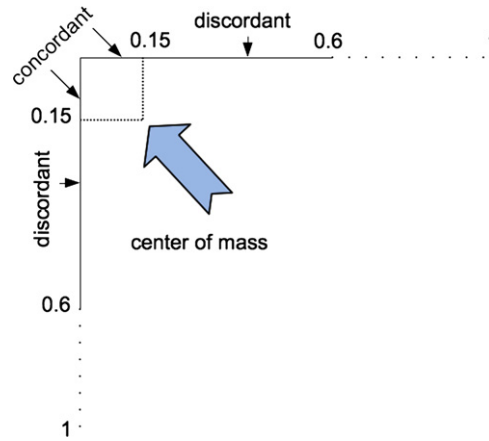
$$M(X_\gamma Y_\gamma) = \lim_{\delta \rightarrow 0} \frac{1}{\int_{[C]_\delta^\gamma} dx dy} \int_{[C]_\delta^\gamma} xy dy dx = 0.$$

Using this we obtain

$$\text{cov}(X_\gamma, Y_\gamma) = -\frac{(1 - \gamma)^2}{16},$$

and for the correlation,

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{3}{5}.$$

Fig. 7. Illustration of  $[C]^{0.4}$ .

Finally we obtain the  $f$ -weighted possibilistic correlation:

$$\rho_f(A, B) = \int_0^1 -\frac{3}{5} f(\gamma) d\gamma = -\frac{3}{5}.$$

In this extremal case, the joint distribution is unequivocally constructed from the knowledge that  $C(x, y) = 0$  for positive  $x, y$ . Now we explain the reason for this negative correlation. Let us choose, for example,  $\gamma = 0.4$  (see Fig. 7). The center of mass of  $[C]^{0.4}$  is  $(0.15, 0.15)$ . The crucial point here is that if we choose any point,  $x$ , from  $[A]^{0.4}$  then the only possible choice from  $[B]^{0.4}$  can be  $y = 0$ , which is always less than 0.15, independently of the choice of  $x$ . In  $[C]^{0.4}$  the strength of discordant points is much bigger than the strength of concordant points, with respect to the reference point  $(0.15, 0.15)$ .

## 6. Question

It is our guess that for these non-symmetrical, but identical marginal distributions,  $A(x) = B(x) = (1 - x)$ , for all  $x \in [0, 1]$ , one cannot define any joint possibility distribution and any  $f$  for which  $\rho_f(A, B)$  could go below the value of  $-3/5$ .

A possibility distribution  $A$  is said to be symmetric if there exists a point  $a \in \mathbb{R}$  such that  $A(a - x) = A(a + x)$  for all  $x \in \mathbb{R}$ . If the membership functions of two symmetrical marginal possibility distributions are equal then we can easily define a joint possibility distribution in such a way that their possibilistic correlation coefficient will be minus one (see Section 5.2). And here comes our question: What is the lower limit for  $f$ -weighted possibilistic correlation coefficient between non-symmetrical marginal possibility distributions with the same membership function?

## 7. Conclusions

We have introduced a novel measure of (relative) index of interactivity between marginal distributions  $A$  and  $B$  of a joint possibility distribution  $C$ . The starting point of our approach is to equip the  $\gamma$ -level set of the joint possibility distribution with a uniform probability distribution. Then the correlation coefficient between its marginal probability distributions is considered to be an index of interactivity between the  $\gamma$ -level sets of  $A$  and  $B$ . If  $[C]^\gamma$  is rectangular for  $0 \leq \gamma < 1$  then  $A$  and  $B$  are non-interactive and their index of interactivity is equal to zero. In the general case we have used the probabilistic correlation coefficient to measure the interactivity between the  $\gamma$ -level sets of  $A$  and  $B$ , which, loosely speaking, measures the ‘strength’ of concordant points as to the ‘strength’ of discordant points of  $[C]^\gamma$  with respect to the center of mass of  $[C]^\gamma$ . This new index of interactivity is meaningful for any joint possibility distribution.

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## Paper 5

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# A Correlation Ratio for Possibility Distributions

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**Abstract.** Generalizing the probabilistic correlation ratio we will introduce a correlation ratio for marginal possibility distributions of joint possibility distributions.

**Keywords:** Correlation ratio, possibility distribution, joint possibility distribution.

## 1 Introduction

In statistics, the correlation ratio is a measure of the relationship between the statistical dispersion within individual categories and the dispersion across the whole population or sample. The correlation ratio was originally introduced by Karl Pearson [5] as part of analysis of variance and it was extended to random variables by Andrei Nikolaevich Kolmogorov [4] as,

$$\eta^2(X|Y) = \frac{D^2[E(X|Y)]}{D^2(X)},$$

where  $X$  and  $Y$  are random variables. If  $X$  and  $Y$  have a joint probability density function, denoted by  $f(x, y)$ , then we can compute  $\eta^2(X|Y)$  using the following formulas

$$E(X|Y = y) = \int_{-\infty}^{\infty} xf(x|y)dx$$

and

$$D^2[E(X|Y)] = E(E(X|y) - E(X))^2,$$

and where,

$$f(x|y) = \frac{f(x, y)}{f(y)}.$$

*Note 1.* The correlation ratio measures the functional dependence between  $X$  and  $Y$ . It takes on values between 0 (no functional dependence) and 1 (purely deterministic dependence). It is worth noting that if  $E(X|Y = y)$  is linear function of  $y$  (i.e. there is a linear relationship between random variables  $E(X|Y)$  and  $Y$ ) this will give the same result as the square of the correlation coefficient, otherwise the correlation ratio will be larger in magnitude. It can therefore be used for judging non-linear relationships. Also note that the correlation ratio is asymmetrical by nature since the two random variables fundamentally do not play the same role in the functional relationship; in general,  $\eta^2(X|Y) \neq \eta^2(Y|X)$ .

A *fuzzy number*.  $A$  is a fuzzy set  $\mathbb{R}$  with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by  $\mathcal{F}$ . Fuzzy numbers can be considered as possibility distributions. A fuzzy set  $C$  in  $\mathbb{R}^2$  is said to be a joint possibility distribution of fuzzy numbers  $A, B \in \mathcal{F}$ , if it satisfies the relationships

$$\max\{x \mid C(x, y)\} = B(y) \quad \text{and} \quad \max\{y \mid C(x, y)\} = A(x)$$

for all  $x, y \in \mathbb{R}$ . Furthermore,  $A$  and  $B$  are called the marginal possibility distributions of  $C$ . A  $\gamma$ -level set (or  $\gamma$ -cut) of a fuzzy number  $A$  is a non-fuzzy set denoted by  $[A]^\gamma$  and defined by  $[A]^\gamma = \{t \in X \mid A(t) \geq \gamma\}$  if  $\gamma > 0$  and  $\text{cl}(\text{supp}A)$  if  $\gamma = 0$ , where  $\text{cl}(\text{supp}A)$  denotes the closure of the support of  $A$ .

Let  $A \in \mathcal{F}$  be fuzzy number with a  $\gamma$ -level set denoted by  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ,  $\gamma \in [0, 1]$  and let  $U_\gamma$  denote a uniform probability distribution on  $[A]^\gamma$ ,  $\gamma \in [0, 1]$ .

In possibility theory we can use the principle of *expected value* of functions on fuzzy sets to define variance, covariance and correlation of possibility distributions. Namely, we can equip each level set of a possibility distribution (represented by a fuzzy number) with a uniform probability distribution, then apply their standard probabilistic calculation, and then define measures on possibility distributions by integrating these weighted probabilistic notions over the set of all membership grades [1,2]. These weights (or importances) can be given by weighting functions. A function  $g: [0, 1] \rightarrow \mathbb{R}$  is said to be a weighting function if  $g$  is non-negative, monotone increasing and satisfies the following normalization condition  $\int_0^1 g(\gamma) d\gamma = 1$ . Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions. In this paper we will introduce a correlation ratio for marginal possibility distributions of joint possibility distributions.

## 2 A Correlation Ratio for Marginal Possibility Distributions

**Definition 1.** Let us denote  $A$  and  $B$  the marginal possibility distributions of a given joint possibility distribution  $C$ . Then the  $g$ -weighted possibilistic correlation ratio of marginal possibility distribution  $A$  with respect to marginal possibility distribution  $B$  is defined by

$$\eta_f^2(A|B) = \int_0^1 \eta^2(X_\gamma|Y_\gamma) g(\gamma) d\gamma \quad (1)$$



where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\eta^2(X_\gamma|Y_\gamma)$  denotes their probabilistic correlation ratio.

So the  $g$ -weighted possibilistic correlation ratio of the fuzzy number  $A$  on  $B$  is nothing else, but the  $g$ -weighted average of the probabilistic correlation ratios  $\eta^2(X_\gamma|Y_\gamma)$  for all  $\gamma \in [0, 1]$ .

### 3 Computation of Correlation Ratio: Some Examples

In this section we will compute the  $g$ -weighted possibilistic correlation ratio for joint possibility distributions  $(1 - x - y)$ ,  $(1 - x^2 - y)$ ,  $(1 - \sqrt{x} - y)$ ,  $(1 - x^2 - y^2)$  and  $(1 - \sqrt{x} - \sqrt{y})$  defined on proper subsets of the unit square.

#### 3.1 A Linear Relationship

Consider the case, when

$$A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by  $C(x, y) = (1 - x - y) \cdot \chi_T(x, y)$ , where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}.$$

Then we have  $[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1 - \gamma\}$ . The density function of a uniform distribution on  $[C]^\gamma$  is

$$f(x, y) = \begin{cases} \frac{2}{(1 - \gamma)^2} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(1 - \gamma - x)}{(1 - \gamma)^2} & \text{if } 0 \leq x \leq 1 - \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{2(1 - \gamma - y)}{(1 - \gamma)^2} & \text{if } 0 \leq y \leq 1 - \gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ration we need to calculate the conditional probability distribution:

$$E(X|Y = y) = \int_0^{1-\gamma-y} x f(x|y) dx = \int_0^{1-\gamma-y} x \frac{f(x, y)}{f_2(y)} dx = \frac{1 - \gamma - y}{2},$$

where  $0 \leq x \leq 1 - \gamma$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(X|Y)] &= E(E(X|y) - E(X))^2 \\ &= \int_0^{1-\gamma} \left( \frac{1-\gamma-y}{2} - \frac{1-\gamma}{3} \right)^2 \frac{2(1-\gamma-y)}{(1-\gamma)^2} \\ &= \frac{(1-\gamma)^2}{72}. \end{aligned}$$

Using the relationship

$$D^2(X_\gamma) = \frac{(1-\gamma)^2}{18},$$

we obtain that the probabilistic correlation of  $X_\gamma$  on  $Y_\gamma$  is

$$\eta^2(X_\gamma|Y_\gamma) = \frac{1}{4}.$$

From this the  $g$ -weighted possibilistic correlation ratio of  $A$  with respect to  $B$  is,

$$\eta_f^2(A|B) = \int_0^1 \frac{1}{4} g(\gamma) d\gamma = \frac{1}{4}.$$

*Note 2.* The  $g$ -weighted normalized measure of interactivity between  $A \in \mathcal{F}$  and  $B \in \mathcal{F}$  (with respect to their joint distribution  $C$ ) is defined by [3]

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) g(\gamma) d\gamma$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}}.$$

and where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\rho(X_\gamma, Y_\gamma)$  denotes their probabilistic correlation coefficient. In this simple case

$$\eta_f^2(A|B) = \eta_f^2(B|A) = [\rho_f(A, B)]^2,$$

since  $E(X_\gamma|Y_\gamma = y)$  is a linear function of  $y$ . Really, in this case we have,

$$\begin{aligned} E(X_\gamma|Y_\gamma = y) &= \frac{1-\gamma-y}{2} = \frac{1-\gamma}{3} - \frac{y}{2} + \frac{1-\gamma}{6} \\ &= \frac{1-\gamma}{3} - \frac{1}{2}y - \left( -\frac{1}{2} \right) \times \frac{1-\gamma}{3} \\ &= \frac{1-\gamma}{3} - \frac{1}{2} \left( y - \frac{1-\gamma}{3} \right) = E(X_\gamma) - \rho(X_\gamma, Y_\gamma)(y - E(Y_\gamma)). \end{aligned}$$

### 3.2 A Nonlinear Relationship

Consider the case, when

$$A(x) = (1 - x^2) \cdot \chi_{[0,1]}(x),$$

$$B(x) = (1 - y) \cdot \chi_{[0,1]}(y),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [0, \sqrt{1-\gamma}]$ ,  $[B]^\gamma = [0, 1-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - x^2 - y) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{2(1-\gamma)^{\frac{3}{2}}} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{3(1-\gamma-x^2)}{2(1-\gamma)^{\frac{3}{2}}} & \text{if } 0 \leq x \leq \sqrt{1-\gamma} \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{3\sqrt{1-\gamma-y}}{2(1-\gamma)^{\frac{3}{2}}} & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ration we need to calculate the conditional probability distribution:

$$E(Y|X=x) = \int_0^{1-\gamma-x^2} y f(y|x) dy = \int_0^{1-\gamma-x^2} y \frac{f(x, y)}{f_1(x)} dy = \frac{1-\gamma-x^2}{2},$$

where  $0 \leq y \leq 1-\gamma$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{\sqrt{1-\gamma}} \left( \frac{1-\gamma-x^2}{2} - \frac{2(1-\gamma)}{5} \right)^2 \frac{3(1-\gamma-x^2)}{2(1-\gamma)^{\frac{3}{2}}} dx \\ &= \frac{2(1-\gamma)^2}{175}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{12(1-\gamma)^2}{175},$$

we obtain that the probabilistic correlation ratio of  $Y_\gamma$  with respect to  $X_\gamma$  is

$$\eta^2(Y_\gamma|X_\gamma) = \frac{1}{6}.$$

From this the  $g$ -weighted possibilistic correlation ratio of  $B$  with respect to  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{1}{6} g(\gamma) d\gamma = \frac{1}{6}.$$

Similarly, from  $D^2[E(X|Y)] = \frac{3(1-\gamma)}{320}$ , and from

$$D^2(X_\gamma) = \frac{19(1-\gamma)}{320},$$

we obtain,

$$\eta_f^2(A|B) = \int_0^1 \frac{3}{19} g(\gamma) d\gamma = \frac{3}{19}.$$

That is  $\eta_f^2(B|A) \neq \eta_f^2(A|B)$ .

### 3.3 Joint Distribution: $(1 - \sqrt{x} - y)$

Consider the case, when

$$A(x) = (1 - \sqrt{x}) \cdot \chi_{[0,1]}(x),$$

$$B(y) = (1 - y) \cdot \chi_{[0,1]}(y),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [0, (1 - \gamma)^2]$ ,  $[B]^\gamma = [0, 1 - \gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - \sqrt{x} - y) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + y \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + y \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{(1 - \gamma)^3} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{3(1-\gamma-\sqrt{x})}{(1-\gamma)^3} & \text{if } 0 \leq x \leq (1-\gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{3(1-\gamma-y)^2}{(1-\gamma)^3} & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ratio we need to calculate the conditional probability distribution:

$$E(Y|X=x) = \int_0^{1-\gamma-\sqrt{x}} y f(y|x) dy = \int_0^{1-\gamma-\sqrt{x}} y \frac{f(x,y)}{f_1(x)} dy = \frac{1-\gamma-\sqrt{x}}{2},$$

where  $0 \leq y \leq 1-\gamma$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{(1-\gamma)^2} \left( \frac{1-\gamma-\sqrt{x}}{2} - \frac{1-\gamma}{4} \right)^2 \frac{3(1-\gamma-\sqrt{x})}{(1-\gamma)^3} dx \\ &= \frac{(1-\gamma)^2}{80}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{3(1-\gamma)^2}{80},$$

we obtain that the probabilistic correlation ratio of  $Y_\gamma$  with respect to  $X_\gamma$  is

$$\eta^2(Y_\gamma|X_\gamma) = \frac{1}{3}.$$

From this the  $g$ -weighted possibilistic correlation ratio of  $B$  with respect to  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{1}{3} g(\gamma) d\gamma = \frac{1}{3}.$$

Similarly, from  $D^2[E(X|Y)] = \frac{3(1-\gamma)^4}{175}$ , and from

$$D^2(X_\gamma) = \frac{37(1-\gamma)^4}{700},$$

we obtain:

$$\eta_f^2(A|B) = \int_0^1 \frac{12}{37} g(\gamma) d\gamma = \frac{12}{37}.$$

### 3.4 A Ball-Shaped Joint Distribution

Consider the case, when

$$A(x) = B(x) = (1 - x^2) \cdot \chi_{[0,1]}(x),$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, \sqrt{1-\gamma}]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is ball-shaped, that is,

$$C(x, y) = (1 - x^2 - y^2) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{4}{(1-\gamma)\pi} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{4\sqrt{1-\gamma-x^2}}{(1-\gamma)\pi} & \text{if } 0 \leq x \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{4\sqrt{1-\gamma-y^2}}{(1-\gamma)\pi} & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ration we need to calculate the conditional probability distribution:

$$E(Y|X=x) = \int_0^{\sqrt{1-\gamma-x^2}} y f(y|x) dy = \int_0^{\sqrt{1-\gamma-x^2}} y \frac{f(x, y)}{f_1(x)} dy = \frac{\sqrt{1-\gamma-x^2}}{2},$$

where  $0 \leq y \leq \sqrt{1-\gamma}$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{\sqrt{1-\gamma}} \left( \frac{\sqrt{1-\gamma-x^2}}{2} - \frac{4\sqrt{1-\gamma}}{3\pi} \right)^2 \frac{4\sqrt{1-\gamma-x^2}}{\pi(1-\gamma)} dx \\ &= \frac{(1-\gamma)(27\pi^2 - 256)}{144\pi^2}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{(1-\gamma)(9\pi^2 - 64)}{36\pi^2},$$

we obtain that the probabilistic correlation ratio of  $Y_\gamma$  with respect to  $X_\gamma$  is

$$\eta^2(Y_\gamma|X_\gamma) = \frac{27\pi^2 - 256}{36\pi^2 - 256}.$$

Finally, we get that the  $g$ -weighted possibilistic correlation ratio of  $B$  with respect  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{27\pi^2 - 256}{36\pi^2 - 256} g(\gamma) d\gamma = \frac{27\pi^2 - 256}{36\pi^2 - 256}.$$

### 3.5 Joint Distribution: $(1 - \sqrt{x} - \sqrt{y})$

Consider the case, when  $A(x) = B(x) = (1 - \sqrt{x}) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, (1 - \gamma)^2]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:

$$C(x, y) = (1 - \sqrt{x} - \sqrt{y}) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1 - \gamma\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{6}{(1 - \gamma)^4} & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{6(1 - \gamma - \sqrt{x})^2}{(1 - \gamma)^4} & \text{if } 0 \leq x \leq (1 - \gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{6(1 - \gamma - \sqrt{y})^2}{(1 - \gamma)^4} & \text{if } 0 \leq y \leq (1 - \gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

For the correlation ration we need to calculate the conditional probability distribution:

$$E(Y|X=x) = \int_0^{(1-\gamma-\sqrt{x})^2} y f(y|x) dy = \int_0^{(1-\gamma-\sqrt{x})^2} y \frac{f(x, y)}{f_1(x)} dy = \frac{(1 - \gamma - \sqrt{x})^2}{2},$$

where  $0 \leq y \leq (1 - \gamma)^2$ . The next step is to calculate the variation of this distribution:

$$\begin{aligned} D^2[E(Y|X)] &= E(E(Y|x) - E(Y))^2 \\ &= \int_0^{(1-\gamma)^2} \left( \frac{(1-\gamma-\sqrt{x})^2}{2} - \frac{(1-\gamma)^2}{5} \right)^2 \frac{6(1-\gamma-\sqrt{x})^2}{(1-\gamma)^4} dx \\ &= \frac{19(1-\gamma)^4}{1400}. \end{aligned}$$

Using the relationship

$$D^2(Y_\gamma) = \frac{9(1-\gamma)^4}{350},$$

we obtain that the probabilistic correlation of  $Y_\gamma$  with respect to  $X_\gamma$  is,

$$\eta^2(Y_\gamma|X_\gamma) = \frac{19}{36}.$$

That is, the  $g$ -weighted possibilistic correlation ratio of  $B$  with respect to  $A$  is,

$$\eta_f^2(B|A) = \int_0^1 \frac{19}{36} g(\gamma) d\gamma = \frac{19}{36}.$$

## 4 Summary

In this paper we have introduced a correlation ratio for marginal possibility distributions of joint possibility distributions. We have illustrated this new principle by five examples.

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## Paper 6

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# Some Examples of Computing the Possibilistic Correlation Coefficient from Joint Possibility Distributions

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**Abstract.** In this paper we will show some examples for computing the possibilistic correlation coefficient between marginal distributions of a joint possibility distribution. First we consider joint possibility distributions,  $(1-x-y)$ ,  $(1-x^2-y^2)$ ,  $(1-\sqrt{x}-\sqrt{y})$  and  $(1-x^2-y)$  on the set  $\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$ , then we will show (i) how the possibilistic correlation coefficient of two linear marginal possibility distributions changes from zero to  $-1/2$ , and from  $-1/2$  to  $-3/5$  by taking out bigger and bigger parts from the level sets of a their joint possibility distribution; (ii) how to compute the autocorrelation coefficient of fuzzy time series with linear fuzzy data.

## 1 Introduction

A fuzzy number  $A$  is a fuzzy set in  $\mathbb{R}$  with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by  $\mathcal{F}$ . Fuzzy numbers can be considered as possibility distributions. A fuzzy set  $C$  in  $\mathbb{R}^2$  is said to be a joint possibility distribution of fuzzy numbers  $A, B \in \mathcal{F}$ , if it satisfies the relationships  $\max\{x \mid C(x; y)\} = B(y)$  and  $\max\{y \mid C(x; y)\} = A(x)$  for all  $x, y \in \mathbb{R}$ . Furthermore,  $A$  and  $B$  are called the marginal possibility distributions of

C. Let  $A \in \mathcal{F}$  be fuzzy number with a  $\gamma$ -level set denoted by  $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ ,  $\gamma \in [0, 1]$  and let  $U_\gamma$  denote a uniform probability distribution on  $[A]^\gamma$ ,  $\gamma \in [0, 1]$ .

In possibility theory we can use the principle of *expected value* of functions on fuzzy sets to define variance, covariance and correlation of possibility distributions. Namely, we equip each level set of a possibility distribution (represented by a fuzzy number) with a uniform probability distribution, then apply their standard probabilistic calculation, and then define measures on possibility distributions by integrating these weighted probabilistic notions over the set of all membership grades. These weights (or importances) can be given by weighting functions. A function  $f: [0, 1] \rightarrow \mathbb{R}$  is said to be a weighting function if  $f$  is non-negative, monotone increasing and satisfies the following normalization condition  $\int_0^1 f(\gamma) d\gamma = 1$ . Different weighting functions can give different (case-dependent) importances to level-sets of possibility distributions.

In 2004 Fullér and Majlender [2] introduced the notion of covariance between marginal distributions of a joint possibility distribution  $C$  as the expected value of their interactivity function on  $C$ . That is, the  $f$ -weighted measure of interactivity between  $A \in \mathcal{F}$  and  $B \in \mathcal{F}$  (with respect to their joint distribution  $C$ ) is defined by their measure of possibilistic covariance [2], as

$$\text{Cov}_f(A, B) = \int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where  $X_\gamma$  and  $Y_\gamma$  are random variables whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ , and  $\text{cov}(X_\gamma, Y_\gamma)$  denotes their probabilistic covariance. They interpreted this covariance as a measure of interactivity between marginal distributions. They also showed that non-interactivity entails zero covariance, however, zero covariance does not always imply non-interactivity. The measure of interactivity is positive (negative) if the expected value of the interactivity relation on  $C$  is positive (negative). It is easy to see that the possibilistic covariance is an absolute measure in the sense that it can take any value from the real line. To have a relative measure of interactivity between marginal distributions we have introduced the normalized covariance in 2010 (see [3]).

**Definition 1.1** ([3]) *The  $f$ -weighted normalized measure of interactivity between  $A \in \mathcal{F}$  and  $B \in \mathcal{F}$  (with respect to their joint distribution  $C$ ) is defined by*

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma, \quad (1)$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)} \sqrt{\text{var}(Y_\gamma)}}$$

Following the terminology of Carlsson, Fullér and Majlender [1] we will call this improved measure of interactivity as the  $f$ -weighted possibilistic correlation ratio.

In other words, the  $f$ -weighted possibilistic correlation coefficient is nothing else, but the  $f$ -weighted average of the probabilistic correlation coefficients  $\rho(X_\gamma, Y_\gamma)$  for all  $\gamma \in [0, 1]$ .

## 2 Some Illustrations of Possibilistic Correlation

### 2.1 Joint Distribution: (1-x-y)

Consider the case, when  $A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, 1-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by  $F(x, y) = (1-x-y) \cdot \chi_T(x, y)$ , where  $T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$ . Then we have  $[F]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1-\gamma\}$ .

This situation is depicted on Fig. 1, where we have shifted the fuzzy sets to get a better view of the situation. In this case the  $f$ -weighted possibilistic correlation of  $A$  and  $B$  is computed as (see [3] for details),

$$\rho_f(A, B) = \int_0^1 -\frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

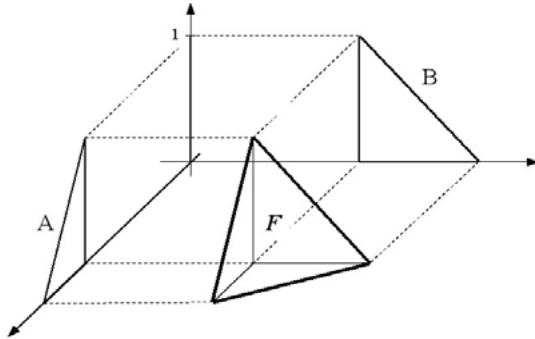


Fig. 1. Illustration of joint possibility distribution  $F$

Consider now the case when  $A(x) = B(x) = x \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [\gamma, 1]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by  $W(x, y) = \max\{x+y-1, 0\}$ . Then we get  $\rho_f(A, B) = -1/2$ . We note here that  $W$  is nothing else but the Lukasiewicz t-norm, or in the statistical literature,  $W$  is generally referred to as the lower Fréchet-Hoeffding bound for copulas.

## 2.2 Joint Distribution: $(1-x^2-y^2)$

Consider the case, when  $A(x) = B(x) = (1-x^2) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, \sqrt{1-\gamma}]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:  $C(x,y) = (1-x^2-y^2) \cdot \chi_T(x,y)$ , where  $T = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2+y^2 \leq 1\}$ . A  $\gamma$ -level set of  $C$  is computed by  $[C]^\gamma = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2+y^2 \leq 1-\gamma\}$ .

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x,y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{4}{(1-\gamma)\pi}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{4\sqrt{1-\gamma-x^2}}{(1-\gamma)\pi}, & \text{if } 0 \leq x \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{4\sqrt{1-\gamma-y^2}}{(1-\gamma)\pi}, & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{4}{(1-\gamma)\pi} \int_0^{\sqrt{1-\gamma}} x \sqrt{1-\gamma-x^2} dx = \frac{4\sqrt{1-\gamma}}{3\pi}$$

$$M(Y_\gamma) = \frac{4}{(1-\gamma)\pi} \int_0^{\sqrt{1-\gamma}} y \sqrt{1-\gamma-y^2} dy = \frac{4\sqrt{1-\gamma}}{3\pi}.$$

We calculate the variations of  $X_\gamma$  and  $Y_\gamma$  with the formula  $\text{var}(X) = M(X^2) - M(X)^2$ :

$$M(X_\gamma^2) = \frac{4}{(1-\gamma)\pi} \int_0^{\sqrt{1-\gamma}} x^2 \sqrt{1-\gamma-x^2} dx = \frac{1-\gamma}{4}$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{1-\gamma}{4} - \frac{16(1-\gamma)}{9\pi^2} = \frac{(1-\gamma)(9\pi^2 - 64)}{36\pi^2}.$$

And similarly we obtain

$$\text{var}(Y_\gamma) = \frac{(1-\gamma)(9\pi^2 - 64)}{36\pi^2}.$$

Using that

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = \frac{(1-\gamma)(9\pi-32)}{18\pi^2},$$

we can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = \frac{2(9\pi-32)}{(9\pi^2-64)} \approx -0.302.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$ :

$$\rho_f(A, B) = \int_0^1 \frac{2(9\pi-32)}{(9\pi^2-64)} f(\gamma) d\gamma = \frac{2(9\pi-32)}{(9\pi^2-64)}.$$

### 2.3 Joint Distribution: $(1-\sqrt{x}-\sqrt{y})$

Consider the case, when  $A(x) = B(x) = (1-\sqrt{x}) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [0, (1-\gamma)^2]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:  $C(x, y) = (1-\sqrt{x}-\sqrt{y}) \cdot \chi_T(x, y)$ , where

$$T = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by  $[C]^\gamma = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, \sqrt{x} + \sqrt{y} \leq 1-\gamma\}$ .

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{6}{(1-\gamma)^4}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{6(1-\gamma-\sqrt{x})^2}{(1-\gamma)^4}, & \text{if } 0 \leq x \leq (1-\gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{6(1-\gamma-\sqrt{y})^2}{(1-\gamma)^4}, & \text{if } 0 \leq y \leq (1-\gamma)^2 \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{6}{(1-\gamma)^4} \int_0^{(1-\gamma)^2} x(1-\gamma-\sqrt{x})^2 dx = \frac{(1-\gamma)^2}{5}$$

$$M(Y_\gamma) = \frac{6}{(1-\gamma)^4} \int_0^{(1-\gamma)^2} y(1-\gamma-\sqrt{y})^2 dy = \frac{(1-\gamma)^2}{5}$$

The variations of  $X_\gamma$  and  $Y_\gamma$  with the formula  $\text{var}(X) = M(X^2) - M(X)^2$ :

$$M(X_\gamma^2) = \frac{6}{(1-\gamma)^4} \int_0^{(1-\gamma)^2} x^2(1-\gamma-\sqrt{x})^2 dx = \frac{(1-\gamma)^4}{14}$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^4}{14} - \frac{(1-\gamma)^4}{25} = \frac{9(1-\gamma)^4}{350}.$$

And similarly we obtain

$$\text{var}(Y_\gamma) = \frac{9(1-\gamma)^4}{350}.$$

Using that

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{13(1-\gamma)^4}{700},$$

we can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{13}{18} \approx -0.722.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$ :

$$\rho_f(A, B) = -\int_0^1 \frac{13}{18} f(\gamma) d\gamma = -\frac{13}{18}.$$



## 2.4 Joint Distribution: $(1-x^2-y)$

Consider the case, when  $A(x) = (1-x^2) \cdot \chi_{[0,1]}(x)$ ,  $B(x) = (1-x) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [0, \sqrt{1-\gamma}]$ ,  $[B]^\gamma = [0, 1-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that their joint possibility distribution is given by:  $C(x,y) = (1-x^2-y) \cdot \chi_T(x,y)$ , where

$$T = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x^2 + y \leq 1\}.$$

A  $\gamma$ -level set of  $C$  is computed by  $[C]^\gamma = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x^2 + y \leq 1-\gamma\}$ . The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x,y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{2(1-\gamma)^{\frac{3}{2}}}, & \text{if } (x,y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{3(1-\gamma-x^2)}{2(1-\gamma)^{\frac{3}{2}}}, & \text{if } 0 \leq x \leq \sqrt{1-\gamma} \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{3\sqrt{1-\gamma-y}}{2(1-\gamma)^{\frac{3}{2}}}, & \text{if } 0 \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{\sqrt{1-\gamma}} x(1-\gamma-x^2) dx = \frac{3\sqrt{1-\gamma}}{8}$$

$$M(Y_\gamma) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{\sqrt{1-\gamma}} y\sqrt{1-\gamma-y} dy = \frac{2(1-\gamma)}{5}.$$

We calculate the variations of  $X_\gamma$  and  $Y_\gamma$ , with the formula  $\text{var}(X) = M(X^2) - M(X)^2$ :

$$M(X_\gamma^2) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{\sqrt{1-\gamma}} x^2(1-\gamma-x^2) dx = \frac{1-\gamma}{5}$$

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{1-\gamma}{5} - \frac{9(1-\gamma)}{64} = \frac{19(1-\gamma)}{320}.$$

$$M(Y_\gamma^2) = \frac{3}{2(1-\gamma)^{\frac{3}{2}}} \int_0^{1-\gamma} y^2 \sqrt{1-\gamma-y} dy = \frac{8(1-\gamma)^2}{35}.$$

$$\text{var}(Y_\gamma) = M(Y_\gamma^2) - M(Y_\gamma)^2 = \frac{8(1-\gamma)^2}{35} - \frac{4(1-\gamma)^2}{25} = \frac{12(1-\gamma)^2}{175}.$$

The covariance of  $X_\gamma$  and  $Y_\gamma$ :

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(1-\gamma)^{\frac{3}{2}}}{40},$$

and we can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\sqrt{\frac{35}{228}} \approx -0.392.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$ :

$$\rho_f(A, B) = \int_0^1 -\sqrt{\frac{35}{228}} f(\gamma) d\gamma = -\sqrt{\frac{35}{228}}.$$

### 3 A Transition from Zero to -1/2

Suppose that a family of joint possibility distribution of  $A$  and  $B$  (where  $A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ ) is defined by

$$C_n(x, y) = \begin{cases} 1 - x - \frac{n-1}{n}y, & \text{if } 0 \leq x \leq 1, x \leq y, \frac{n-1}{n}y + x \leq 1 \\ 1 - \frac{n-1}{n}x - y, & \text{if } 0 \leq y \leq 1, y \leq x, \frac{n-1}{n}x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

In the following, for simplicity, we will write  $C$  instead of  $C_n$ . A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{n}{2n-1}(1-\gamma), 0 \leq y \leq 1-\gamma - \frac{n-1}{n}x \right\} \cup \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{n}{2n-1}(1-\gamma) \leq x \leq 1-\gamma, 0 \leq \frac{n-1}{n}y \leq 1-\gamma-x \right\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2n-1}{n(1-\gamma)^2}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the marginal density functions:

$$f_1(x) = \begin{cases} \frac{(2n-1)(1-\gamma-x)}{(n-1)(1-\gamma)^2}, & \text{if } \frac{n}{2n-1}(1-\gamma) \leq x \leq 1-\gamma \\ \frac{(2n-1)(1-\gamma-\frac{n-1}{n}x)}{n(1-\gamma)^2}, & \text{if } 0 \leq x \leq \frac{n}{2n-1}(1-\gamma) \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_2(y) = \begin{cases} \frac{(2n-1)(1-\gamma-y)}{(n-1)(1-\gamma)^2}, & \text{if } \frac{n}{2n-1}(1-\gamma) \leq y \leq 1-\gamma \\ \frac{(2n-1)(1-\gamma-\frac{n-1}{n}y)}{n(1-\gamma)^2}, & \text{if } 0 \leq y \leq \frac{n}{2n-1}(1-\gamma) \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$  as,

$$\begin{aligned} M(X_\gamma) &= \frac{2n-1}{n(1-\gamma)^2} \int_0^{\frac{n(1-\gamma)}{2n-1}} x(1-\gamma-\frac{n-1}{n}x)dx \\ &+ \frac{2n-1}{(n-1)(1-\gamma)^2} \int_{\frac{n(1-\gamma)}{2n-1}}^{1-\gamma} x(1-\gamma-x)dx = \frac{(1-\gamma)(4n-1)}{6(2n-1)} \end{aligned}$$

and

$$M(Y_\gamma) = \frac{(1-\gamma)(4n-1)}{6(2n-1)}.$$

(We can easily see that for  $n = 1$  we have  $M(X_\gamma) = \frac{1-\gamma}{2}$ , and for  $n \rightarrow \infty$  we find  $M(X_\gamma) \rightarrow \frac{1-\gamma}{2}$ .)

We calculate the variations of  $X_\gamma$  and  $Y_\gamma$  as,

$$\begin{aligned} M(X_\gamma^2) &= \frac{2n-1}{n(1-\gamma)^2} \int_0^{\frac{n(1-\gamma)}{2n-1}} x^2(1-\gamma-\frac{n-1}{n}x)dx \\ &+ \frac{2n-1}{(n-1)(1-\gamma)^2} \int_{\frac{n(1-\gamma)}{2n-1}}^{1-\gamma} x^2(1-\gamma-x)dx \\ &= \frac{(1-\gamma)^2((2n-1)^3 + 8n^3 - 6n^2 + n)}{12(2n-1)^3}. \end{aligned}$$

(We can easily see that for  $n = 1$  we have  $M(X_\gamma^2) = \frac{(1-\gamma)^2}{3}$ , and for  $n \rightarrow \infty$  we find  $M(X_\gamma^2) \rightarrow \frac{(1-\gamma)^2}{6}$ .)

Furthermore,

$$\begin{aligned}\text{var}(X_\gamma) &= M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^2((2n-1)^3 + 8n^3 - 6n^2 + n)}{12(2n-1)^3} \\ &\quad - \frac{(1-\gamma)^2(4n-1)^2}{36(2n-1)^2} = \frac{(1-\gamma)^2(2(2n-1)^2 + n)}{36(2n-1)^2}.\end{aligned}$$

And similarly we obtain

$$\text{var}(Y_\gamma) = \frac{(1-\gamma)^2(2(2n-1)^2 + n)}{36(2n-1)^2}.$$

(We can easily see that for  $n = 1$  we have  $\text{var}(X_\gamma) = \frac{(1-\gamma)^2}{12}$ , and for  $n \rightarrow \infty$  we find  $\text{var}(X_\gamma^2) \rightarrow \frac{(1-\gamma)^2}{18}$ .)

And,

$$\begin{aligned}\text{cov}(X_\gamma, Y_\gamma) &= M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) \\ &= \frac{(1-\gamma)^2 n(4n-1)}{12(2n-1)^2} - \frac{(1-\gamma)^2(1-n)(4n-1)}{36(2n-1)^2}.\end{aligned}$$

(We can easily see that for  $n = 1$  we have  $\text{cov}(X_\gamma, Y_\gamma) = 0$ , and for  $n \rightarrow \infty$  we find  $\text{cov}(X_\gamma, Y_\gamma) \rightarrow -\frac{(1-\gamma)^2}{36}$ .)

We can calculate the probabilistic correlation of the random variables,

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = \frac{(1-n)(4n-1)}{2(2n-1)^2 + n}.$$

(We can easily see that for  $n = 1$  we have  $\rho(X_\gamma, Y_\gamma) = 0$ , and for  $n \rightarrow \infty$  we find  $\rho(X_\gamma, Y_\gamma) \rightarrow -\frac{1}{2}$ .)

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$  is computed as,

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma = \frac{(1-n)(4n-1)}{2(2n-1)^2 + n}.$$

We obtain, that  $\rho_f(A, B) = 0$  for  $n = 1$  and if  $n \rightarrow \infty$  then  $\rho_f(A, B) \rightarrow -\frac{1}{2}$ .

#### 4 A Transition from -1/2 to -3/5

Suppose that a family of joint possibility distribution of  $A$  and  $B$  (where  $A(x) = B(x) = (1-x) \cdot \chi_{[0,1]}(x)$ , for  $x \in \mathbb{R}$ ) is defined by

$$C_n(x, y) = (1 - x - y) \cdot \chi_{T_n}(x, y),$$

where

$$T_n = \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1, \frac{1}{n-1}x \geq y \right\} \cup \\ \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1, (n-1)x \leq y \right\}.$$

In the following, for simplicity, we will write  $C$  instead of  $C_n$ . A  $\gamma$ -level set of  $C$  is computed by

$$[C]^\gamma = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{1}{n}(1 - \gamma), (n-1)x \leq y \leq 1 - \gamma - x \right\} \cup \\ \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{1}{n}(1 - \gamma), (n-1)y \leq x \leq 1 - \gamma - y \right\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{n}{(1 - \gamma)^2}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the marginal density functions:

$$f_1(x) = \begin{cases} \frac{n(1 - \gamma - nx + \frac{x}{n-1})}{(1 - \gamma)^2}, & \text{if } 0 \leq x \leq \frac{1 - \gamma}{n} \\ \frac{nx}{(1 - \gamma)^2(n-1)}, & \text{if } \frac{(1 - \gamma)}{n} \leq x \leq \frac{(n-1)(1 - \gamma)}{n} \\ \frac{n(1 - \gamma - x)}{(1 - \gamma)^2}, & \text{if } \frac{(n-1)(1 - \gamma)}{n} \leq x \leq 1 - \gamma \\ 0 & \text{otherwise} \end{cases}$$

and,

$$f_2(y) = \begin{cases} \frac{n(1-\gamma-ny+\frac{y}{n-1})}{(1-\gamma)^2}, & \text{if } 0 \leq y \leq \frac{1-\gamma}{n} \\ \frac{ny}{(1-\gamma)^2(n-1)}, & \text{if } \frac{(1-\gamma)}{n} \leq y \leq \frac{(n-1)(1-\gamma)}{n} \\ \frac{n(1-\gamma-y)}{(1-\gamma)^2}, & \text{if } \frac{(n-1)(1-\gamma)}{n} \leq y \leq 1-\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$  as,

$$\begin{aligned} M(X_\gamma) &= \frac{n}{(1-\gamma)^2} \int_0^{\frac{1-\gamma}{n}} x(1-\gamma-nx+\frac{x}{n-1})dx \\ &\quad + \frac{n}{(1-\gamma)^2} \int_{\frac{1-\gamma}{n}}^{\frac{(n-1)(1-\gamma)}{n}} \frac{x^2}{n-1}dx + \frac{n}{(1-\gamma)^2} \int_{\frac{(n-1)(1-\gamma)}{n}}^{1-\gamma} x(1-\gamma-x)dx \\ &= \frac{1-\gamma}{3}. \end{aligned}$$

That is,  $M(Y_\gamma) = \frac{1-\gamma}{3}$ . We calculate the variations of  $X_\gamma$  and  $Y_\gamma$  as,

$$\begin{aligned} M(X_\gamma^2) &= \frac{n}{(1-\gamma)^2} \int_0^{\frac{1-\gamma}{n}} x^2(1-\gamma-nx+\frac{x}{n-1})dx \\ &\quad + \frac{n}{(1-\gamma)^2} \int_{\frac{1-\gamma}{n}}^{\frac{(n-1)(1-\gamma)}{n}} \frac{x^3}{n-1}dx \\ &\quad + \frac{n}{(1-\gamma)^2} \int_{\frac{(n-1)(1-\gamma)}{n}}^{1-\gamma} x^2(1-\gamma-x)dx \\ &= \frac{(1-\gamma)^2(3n^2-3n+2)}{12n^2}. \end{aligned}$$

and,

$$\begin{aligned} \text{var}(X_\gamma) &= M(X_\gamma^2) - M(X_\gamma)^2 = \frac{(1-\gamma)^2(3n^2-3n+2)}{12n^2} - \frac{(1-\gamma)^2}{9} \\ &= \frac{(1-\gamma)^2(5n^2-9n+6)}{36n^2}. \end{aligned}$$

And, similarly, we obtain

$$\text{var}(Y_\gamma) = \frac{(1-\gamma)^2(5n^2-9n+6)}{36n^2}.$$

From

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = \frac{(1-\gamma)^2(3n-2)}{12n^2} - \frac{(1-\gamma)^2}{9},$$

we can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{-3n^2 + 7n - 6}{5n^2 - 9n + 6}.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$ :

$$\rho_f(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) f(\gamma) d\gamma = -\frac{-3n^2 + 7n - 6}{5n^2 - 9n + 6}$$

We obtain, that for  $n = 2$

$$\rho_f(A, B) = -\frac{1}{2}$$

and if  $n \rightarrow \infty$  then

$$\rho_f(A, B) \rightarrow -\frac{3}{5}$$

We note that in this extreme case the joint possibility distribution is nothing else but the marginal distributions themselves, that is,  $C_\infty(x, y) = 0$ , for any interior point  $(x, y)$  of the unit square.

## 5 Trapezoidal Marginal Distributions

Consider now the case,

$$A(x) = B(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } 1 \leq x \leq 2 \\ 3-x, & \text{if } 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

for  $x \in \mathbb{R}$ , that is  $[A]^\gamma = [B]^\gamma = [\gamma, 3-\gamma]$ , for  $\gamma \in [0, 1]$ . Suppose that the joint possibility distribution of these two trapezoidal marginal distributions – a considerably truncated pyramid – given by:

$$C(x, y) = \begin{cases} y, & \text{if } 0 \leq x \leq 3, 0 \leq y \leq 1, x \leq y, x \leq 3-y \\ 1, & \text{if } 1 \leq x \leq 2, 1 \leq y \leq 2, x \leq y \\ x, & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 3, y \leq x, x \leq 3-y \\ 0, & \text{otherwise} \end{cases}$$

Then  $[C]^\gamma = \{(x, y) \in \mathbb{R}^2 \mid \gamma \leq x \leq 3-\gamma, \gamma \leq y \leq 3-x\}$ . The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{(3-2\gamma)^2}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(3-\gamma-x)}{(3-2\gamma)^2}, & \text{if } \gamma \leq x \leq 3-\gamma \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_2(y) = \begin{cases} \frac{2(3-\gamma-y)}{(3-2\gamma)^2}, & \text{if } \gamma \leq y \leq 3-\gamma \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} x(3-\gamma-x) dx = \frac{\gamma+3}{3}$$

and,

$$M(Y_\gamma) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} y(3-\gamma-y) dy = \frac{\gamma+3}{3}$$

We calculate the variations of  $X_\gamma$  and  $Y_\gamma$ , with the formula  $\text{var}(X) = M(X^2) - M(X)^2$ :

$$M(X_\gamma^2) = \frac{2}{(3-2\gamma)^2} \int_\gamma^{3-\gamma} x^2(3-\gamma-x) dx = \frac{2\gamma^2+9}{6}$$

and,

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{2\gamma^2+9}{6} - \frac{(\gamma+3)^2}{9} = \frac{(3-2\gamma)^2}{18}.$$

And similarly we obtain

$$\text{var}(Y_\gamma) = \frac{(3-2\gamma)^2}{18}.$$

Using the relationship,

$$\text{cov}(X_\gamma, Y_\gamma) = M(X_\gamma Y_\gamma) - M(X_\gamma)M(Y_\gamma) = -\frac{(3-2\gamma)^2}{36},$$



we can calculate the probabilistic correlation of the random variables:

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

And finally the  $f$ -weighted possibilistic correlation of  $A$  and  $B$  is,

$$\rho_f(A, B) = -\int_0^1 \frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

## 6 Time Series with Fuzzy Data

A time series with fuzzy data is referred to as fuzzy time series (see [4]). Consider a fuzzy time series indexed by  $t \in (0, 1]$ :

$$A_t(x) = \begin{cases} 1 - \frac{x}{t}, & \text{if } 0 \leq x \leq t \\ 0 & \text{otherwise} \end{cases}$$

and

$$A_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that in this case,  $[A_t]^\gamma = [0, t(1-\gamma)]$ , for  $\gamma \in [0, 1]$ . If we have  $t_1, t_2 \in (0, 1]$ , then the joint possibility distribution of the corresponding fuzzy numbers is given by:

$$C(x, y) = \left(1 - \frac{x}{t_1} - \frac{y}{t_2}\right) \cdot \chi_T(x, y),$$

where

$$T = \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \frac{x}{t_1} + \frac{y}{t_2} \leq 1 \right\}.$$

Then

$$[C]^\gamma = \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \frac{x}{t_1} + \frac{y}{t_2} \leq 1 - \gamma \right\}.$$

The density function of a uniform distribution on  $[C]^\gamma$  can be written as

$$f(x, y) = \begin{cases} \frac{1}{\int_{[C]^\gamma} dx dy}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

That is,

$$f(x, y) = \begin{cases} \frac{2}{t_1 t_2 (1 - \gamma)^2}, & \text{if } (x, y) \in [C]^\gamma \\ 0 & \text{otherwise} \end{cases}$$

The marginal functions are obtained as

$$f_1(x) = \begin{cases} \frac{2(1 - \gamma - \frac{x}{t_1})}{t_1(1 - \gamma)^2}, & \text{if } 0 \leq x \leq t_1(1 - \gamma) \\ 0 & \text{otherwise} \end{cases}$$

and,

$$f_2(y) = \begin{cases} \frac{2(1 - \gamma - \frac{y}{t_2})}{t_2(1 - \gamma)^2}, & \text{if } 0 \leq y \leq t_2(1 - \gamma) \\ 0 & \text{otherwise} \end{cases}$$

We can calculate the probabilistic expected values of the random variables  $X_\gamma$  and  $Y_\gamma$ , whose joint distribution is uniform on  $[C]^\gamma$  for all  $\gamma \in [0, 1]$ :

$$M(X_\gamma) = \frac{2}{t_1(1 - \gamma)^2} \int_0^{t_1(1 - \gamma)} x(1 - \gamma - \frac{x}{t_1}) dx = \frac{t_1(1 - \gamma)}{3}$$

and

$$M(Y_\gamma) = \frac{2}{t_2(1 - \gamma)^2} \int_0^{t_2(1 - \gamma)} y(1 - \gamma - \frac{y}{t_2}) dy = \frac{t_2(1 - \gamma)}{3}.$$

We calculate now the variations of  $X_\gamma$  and  $Y_\gamma$  as,

$$M(X_\gamma^2) = \frac{2}{t_1(1 - \gamma)^2} \int_0^{t_1(1 - \gamma)} x^2(1 - \gamma - \frac{x}{t_1}) dx = \frac{t_1^2(1 - \gamma)^2}{6}$$

and,

$$\text{var}(X_\gamma) = M(X_\gamma^2) - M(X_\gamma)^2 = \frac{t_1^2(1 - \gamma)^2}{6} - \frac{t_1^2(1 - \gamma)^2}{9} = \frac{t_1^2(1 - \gamma)^2}{18}.$$

And, in a similar way, we obtain,

$$\text{var}(Y_\gamma) = \frac{t_2^2(1-\gamma)^2}{18}.$$

From,

$$\text{cov}(X_\gamma, Y_\gamma) = -\frac{t_1 t_2 (1-\gamma)^2}{36}.$$

we can calculate the probabilistic correlation of the random variables,

$$\rho(X_\gamma, Y_\gamma) = \frac{\text{cov}(X_\gamma, Y_\gamma)}{\sqrt{\text{var}(X_\gamma)}\sqrt{\text{var}(Y_\gamma)}} = -\frac{1}{2}.$$

The  $f$ -weighted possibilistic correlation of  $A_{t_1}$  and  $A_{t_2}$ ,

$$\rho_f(A_{t_1}, A_{t_2}) = \int_0^1 -\frac{1}{2} f(\gamma) d\gamma = -\frac{1}{2}.$$

So, the autocorrelation function of this fuzzy time series is constant. Namely,

$$R(t_1, t_2) = -\frac{1}{2}$$

for all  $t_1, t_2 \in [0, 1]$ .

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